Business Mathematics

Business Mathematics

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BCIT

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Introduction

OBJECTIVE

This Business Mathematics courses aim to have students understand basic business problems, interpret them in mathematical terms, and use the tools of mathematics to solve these problems. The objective of this material is to provide mathematical theory, demonstration and practice for first-year Business students.

A grade of C+ or better in Algebra 11 (or equivalent) is the prerequisite for this course; students requiring extensive review or remedial work are expected to have completed a preparatory course such as OPMT 0199, which is offered through part-time studies at BCIT.

These notes explain the applications of basic mathematics to business and industry using ratios, functions and graphs, simple and compound interest, financial instruments and discounting, annuities, mortgages, loans, and leases. Cash-flow analysis applying rates of return, net present value, and payback is included.

We expect students to read the material carefully and follow the worked examples. You should also work through the learning activities in each chapter and check the recommended solutions. Practice problem sets with selected answers are included at the end of each chapter. You are expected to have a pre-programmed business/financial calculator; the Texas Instruments BA II Plus is recommended, and will be used in solving problems.

CALCULATOR INSTRUCTIONS

The recommended calculator for this course is the Texas Instruments BAII Plus. You may purchase another calculator as long as that calculator has per-programmed financial functions. Your calculator should be able to perform **Net Present Value** (NPV) and **Internal Rate of Return (IRR**) cash flow analysis. Please talk to the instructor before purchasing any calculator other than the BAII plus.

Formatting a BAII Plus:

Step	To	Press	Display
1	Select FORMAT function	[2ND] [./FORMAT]	DEC= 2.00
2	Change the number of decimals to 9 *(with 9 decimals it hides the zeros after the decimal point)*	[9][ENTER]	**DEC= 9**
3	Change calculator to BEDMAS (**AOS**) (gives correct order of operations) Scroll to where it says AOS or Chn ** You want **AOS** Check by entering: $3+2\times 5$ You should get 13 (not 25)	[↓] [↓] [↓] [↓] [2ND][ENTER/SET]	DEG US-12-31-1990 US 1,000 Chn *Skip step if it shows AOS* **AOS**
4	To get out of the FORMAT function	[2ND][CPT/ QUIT]	**0**

Or if you are a visual learner, here's a handy video, courtesy of former BCIT instructor Chris Kellman: Setting up your BAII Plus

Another option:

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://pressbooks.bccampus.ca/businessmathematics/?p=4#video-4-1

Memory Operations with the BAII Plus

Compute: $$\frac{525+287+309}{\left(\frac{0.4}{0.6}\right)}$$

Step	**To**	**Press**	**Display**
1	Sum $525 + 287 + 309$	[5][2][5][+][2][8][7][+][3][0][9][=]	1,121
2	Store the value in Memory 1	[STO][1]	
3	0.4 divided by 0.6	[.][4][÷][.][6][=]	0.666666667
2	Store the value in Memory 2	[STO][2]	
5	Memory 1 divided by Memory 2	[RCL][1][÷][RCL][2][=]	1681.5

Acknowledgements

BUSINESS MATHEMATICS

Notes for courses in the Mathematics of Business and Finance

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- close up photo of assorted coins by Josh Appel available under the Unsplash License

Accessibility Statement

We have tried to the best of our ability to ensure that this textbook is as accessible as possible. If there are any issues, please contact us at Business_MAth@BCIT.ca, and we will try to fix.

The online version of this text contains videos, which are not in the PDF version. You can view these at pressbooks.bccampus.ca/businessmathematics.The videos are additional, and not necessary for the understanding of this material. They have been auto-captioned by YouTube, but there may be some errors.

The online version of this text includes some interactive activities. The PDF gives an address where each activity can be found.

Chapter 1: Business Applications of Basic Mathematics

The purpose of a business is to make a profit. To find the profit a company makes in a particular period, say in a given month, you must start with all the money that comes into the company as a result of its operations. Then from that money, which is called the revenue, subtract all the expenses paid to earn it.

Thus you have

Revenue
– *Expenses*
= *Profit*

The profit is found on the famous “bottom line.” In accounting it is usually called the *net income.*

Example 1.0

Suppose that a company’s revenue is earned entirely from sales and that it had expenses of $24,000 in a period during which it had sales of $32,000.

Then,

	Revenue	**$32,000**
–	**Expenses**	**$24,000**
=	**Profit**	**$8,000**

REVENUE: $32,000

Expenses	**Profit**
$24,000	$8,000

When you study accounting, you will find that it takes considerable effort to find the correct values for the expenses that apply to the revenues for a period.

1.1 Ratios

This chapter uses ratios to analyze basic business concepts such as **profits**, **expenses** and **revenues**.

Key Takeaways

If only two quantities are involved, they are referred to as amount and base.

A ratio expresses the relative sizes of quantities. If only two quantities are involved, they are referred to as *amount* and *base*.

$$Ratio = \frac{Amount}{Base}$$

A frequently used ratio is the ratio of profit to sales. In this case the base is sales and the amount is profit.

Thus you have

$$Ratio = \frac{Amount}{Base} = \frac{Profit}{Sales} = \frac{\$8,000}{\$32,000} = 0.25 = 25\%$$

This ratio is also written as

$$Amount : Base = \$8,000 : \$32,000 = 1 : 4 = 0.25 : 1$$

Key Takeaways

It is common to express a pure ratio in percent.

Since both the amount and base are in dollars, this is a pure ratio or fraction, and it is common in business to express it in percent.

Percent is a ratio in which the base is 100. Thus, for the above example,

$$Ratio = 0.25 = \frac{25}{100} = 25\%$$

where the symbol % signifies that the amount 25 is to be divided by 100.

A ratio of 30% is thus:

$$30\% = \frac{30}{100} = 0.30$$

Percent is used so much in business analysis that you must become completely familiar with its meaning and be able to change back and forth between percent and decimals.

The profit-to-sales ratio expressed this way is called the ***percent net margin***. It is used to compare the profitability of different companies or branches of companies.

Example 1.1.1

Consider a company that has three branches, A, B and C, operating in small, medium-sized and large towns. Suppose the branches have the following performances during a month:

	A	**B**	**C**	**Total**
Sales	\$12,500	\$46,500	\$63,200	\$122,200
Expenses	\$9,550	\$38,750	\$48,850	\$97,150
Profit	\$2,950	\$7,750	\$14,350	\$25,050

From these raw figures it is not easy to assess the performance of the branches. It is clear that C's profit is greater than B's, and that B's is greater than A's, but you would expect that to occur because of the size of their markets and the volume of their sales. If you take the percent net margin, however, you get the following:

$$\text{For A: } \frac{\$2,950}{\$12,500} = 0.236 = 23.6\%$$

For all branches and the company total:

	A	**B**	**C**	**TOTAL**
Percent Net Margin	23.6%	16.7%	22.7%	20.5%

From those percentages, you can see that, in a sense, branch A has the best performance (for its size), and that branch B is farthest out of place. Although there may be good reasons for this, the ratios provide a reasonable place to start looking for improvements.

Examining profits on graphs illustrates the situation. First look at the graph of the dollar values of the profits (Figure 1.1).

Figure 1.1: Profits in Dollars

Then look at the graph of profits as a percent of sales (Figure 1.2 below).

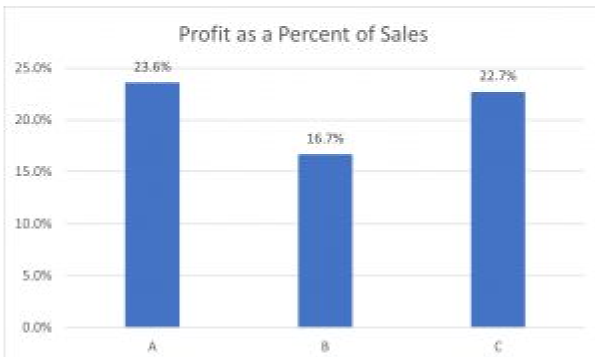

Figure 1.2: Profits as a Percent of Sales

On the last graph you can see that branch B is noticeably lower.

Ratios are frequently used in business analysis. Two examples:

- ***Turnover ratio*** -This is the value (at cost) of goods sold, divided by the average value of the goods held for sale (inventory). This gives the number of times the inventory is “turned over” (completely sold and replaced).

- ***Acid test ratio*** – This is the ratio of the cash resources (e.g., cash and customer accounts owed to the company) to the short-term debts of the company. It is a measure of the ability of a firm to pay its short-term debts.

Knowledge Check 1.1

1. Central Clothing Store had sales of $80,000 last month.

   a. If the total expenses were $71,000, what was the profit?
   b. What must expenses have been if the profit had been $14,000?

2. Two consumer electronics stores, Sounders and Electronicks, are to be evaluated. Their sales for last year were as follows:

	Sounders	Electronicks
Value of goods sold (at cost)	$936,400	$1,245,900

On average, Sounders stock on hand cost them $184,000. Electronicks stock on hand cost them $209,400. Compare their turnover ratios. Which had the better (higher) ratio?

Your Own Notes

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https://pressbooks.bccampus.ca/businessmathematics/?p=52#h5p-1

1.2 Proportions

Another way to analyze the performance of branches A, B, and C described above is to take the view that all of the percent margins should be equal – or, equivalently, that the ratio of profit to sales should be the same for each branch.

When the ratio of quantities should be the same in all circumstances, the quantities are said to be *proportional* to one another.

Example 1.2.1

Assume that the profit should be proportional to sales, and that this proportion should be that of the company totals (20.5%). Then you could calculate the appropriate profits for branches A, B and C individually.

For branch A:

$$Ratio = \frac{20.5}{100} = 0.205$$

$$0.205 = \frac{Amount}{Base} = \frac{Profit}{\$12,500}$$

Thus Profit = $0.205 \times \$12,500 = \$2,562.50$

Similarly for branches B and C, the proportional profits would be $9,532.50 and $12,956 respectively. The analysis would proceed as follows:

Profits	A	B	C
Actual	$2,950.00	$7,750.00	$14,350.00
Proportional	2,562.50	9,532.50	12,956 .00
Difference	+387.50	-1,782.50	+1,394.00

Again you can see the branch B's performance was the least effective.

Ratios are generally used as in the above exercises – sometimes to eliminate the effect of size, and sometimes "in reverse" to make sure the quantities are appropriate.

Knowledge Check 1.2

1. Will's Coffee Shops expect to earn a percent net margin of 14% of sales.
    a. Find the expected profit and expenses of a shop that will have sales of $29,000 a month.
    b. If a shop hopes to earn a net profit of $5,000 a month, what should be its sales target?
2. As a guide to restaurant pricing, it is often assumed that the price of a meal should be proportional to the cost of the food in the meal. Suppose that a restaurant decided to make that ratio 2.8.

That is:

$$\frac{\text{Meal Price}}{\text{Food Price}} = 2.8$$

Use this guide to estimate the price of meals for which the food cost is (a) $3.50; (b) $8.00; (c) $14.00.

Your Own Notes

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1.3 Ratios With More Than Two Quantities

In some business situations, it is desirable to examine the relative sizes of a number of quantities. In such cases, ratios of the quantities are used.

Consider, for example, the expenses of a merchandising company. These expenses can be usefully broken down into two major components:

- the **cost of the goods sold (COGS)**
- all other expenses, called operating expenses.

When analyzing merchandising companies, it is customary to first deduct the cost of goods sold from sales revenue, and to call the result the gross profit.

$$Sales - COGS = Gross\ Profit$$

Operating expenses are then deducted from the gross profit to get net profit.

SALES
– COGS
GROSS PROFIT
– OPERATING EXPENSES
NET PROFIT

All of these quantities are commonly compared with the sales revenue as the base.

Example 1.3.1

Suppose that in our opening example, in which the company had sales of $32,000 in a period, the cost of these goods was $20,000 and operating expenses were $4,000. You would arrange the analysis as follows:

	$	% Sales
Sales	$32,000	100.00
– COGS	20,000	62.5
Gross Profit	12,000	37.5
– Operating Expenses	4,000	12.5
Net Profit	8,000	25.0

This has the format of a (very) simple Income Statement.

When gross profit is compared with sales it is called **percent gross margin**. Net profit compared with sales is called **percent net margin**.

Key Takeaways

*Gross profit compared with sales is called **percent gross margin**; Net profit compared with sales is called **percent net margin**.*

Suppose now that a similar company was expected to perform with the same ratios but to have sales of \$50,000 in a period. Then each of the other entries could be calculated by using the above ratios. For example:

$$\frac{COGS}{Sales} = \frac{COGS}{\$50,000} = \frac{62.5}{100} = 0.625$$

So

$$COGS = Sales \times 0.625 = \$50,000 \times 0.625 = \$31,250$$

Notice that while the ratio was stated as a percent, it was necessary to replace it with the decimal equivalent in order to do the calculations. You should check that the other quantities come out, as in the following table:

		% **Sales**
Sales	\$50,000	100.0
COGS	31,250	62.5
Gross Profit	18,750	37.5
Operating Expenses	6,250	12.5
Net Profit	12,500	25.0

Another use of ratios is to allocate resources according to some measure of need or performance in an organization. For example, bonuses may be given as a percent of sales; budgeted expenses may be a percent of the previous year's expenses.

Knowledge Check 1.3

A merchandising company finds that its COGS is 65% of sales and its monthly operating expenses are $14,000.

What would be the COGS and the gross and net profits on sales of $90,000 in a month?

What would be its percent gross margin and percent net margin?

Your Own Notes

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1.4 Retail Calculations

Many retail calculations parallel the foregoing ratio calculations for profits and expenses.

When goods are bought for resale, a decision has to be made about the selling price of the individual articles. The goods are said to be *marked up*, the amount added to the cost of the goods being the ***markup***.

Thus

$$\text{Selling Price} = \text{Cost} + \text{Markup}$$

or

$$S = C + M$$

if S is used for selling price, C for cost and M for markup.

Key Takeaway

In setting markup, it is common to apply the same fraction of the cost for marking up all goods in the same category.

This is similar to the calculation of gross profit for a company:

	COMPANY		ARTICLE
	SALES		SELLING PRICE
–	COGS	–	COST
	GROSS PROFIT		GROSS PROFIT (MARKUP)

To simplify the process of finding markup for each product, it is common to apply the same fraction of the cost as that used for marking up all goods in the same category – for example, sports shoes.

$$Markup\ Ratio = \frac{Amount\ of\ Markup}{Cost\ of\ Goods}$$

This ratio is usually written as a percent and called ***percent markup***.

$$\%Markup = \frac{Markup}{Cost} \times 100\%$$

For a markup ratio of 25% and goods that cost $20.00 per unit.

Key Takeaway

Gross profit: Markup ratio times cost of goods.

So

$$\text{Amount of Markup} = (\text{Markup ratio}) \times (\text{Cost of Goods}) = 0.25 \times \$20.00 = \$5.00$$

And the selling price would be

$$S = C + M = \$20.00 + \$5.00 = \$25.00$$

The $5.00 is also called the ***gross profit*** on the sale.

Note that, **in general**, markup ratios are based on cost prices and that margin ratios are based on sales or selling price. For the above example, if you wanted a margin ratio for the product, you would compare the $5.00 to the selling price of $25.00 and get

$$\text{Percent Gross Margin} = \frac{\$5}{\$25} = 0.20 = 20\%$$

Since the percent markup is based on cost, which is known at the time goods are purchased, the normal way to calculate the increase is to use the percent markup method.

Careful!

Even though this book stays with the definitions of margin and markup as above, you should be aware that some texts and industries may use different terms (for example, “markon” instead of “markup”, or “markup on sales” for margin) and different meanings for the terms. But the ones used here are frequently used, and you will find them on most business calculators.

Knowledge Check 1.4

1. Complete the following table.

	COST	%MARKUP	MARKUP	PRICE
a.	$10.00	35%	?	?
b.	$20.00	?	$9.00	?
c.	$16.00	?	?	$20.00

Your Own Notes

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1.5 Rates and Currency Conversions

Key Takeaway

Rate: The result when you take the ratio of two different units of measurement.

When you take the ratio of two quantities that have different units of measurement (for example, money and time), you refer to the resulting ratio as a *rate*. As before, using the terms amount and the base:

$$Rate = \frac{Amount}{Base}$$

A sales rate would have the amount in dollars and the base, perhaps, in days. So if sales were $18,000 in a 10-day period, you would have:

$$\text{Sales Rate} = \frac{\$18,000}{10 \text{ days}} = \$1,800 \text{ per day}$$

Similarly a pay rate might be in dollars per hour and an exchange rate in dollars per British pound.

One use of such rates is to find a new amount for a different base (assuming the rate is constant). You would have to do this if you wanted to find the cost of a given amount of foreign money. In this case, you would check the exchange rates, which are determined by money markets and reported regularly in the financial pages of newspapers.

Example 1.5.1

Suppose the exchange rate is $2.10 per British pound and you need £300. Then:

$$Rate = \frac{\text{Amount in } \$}{\text{Amount in } £} = \frac{\$2.10}{£1}$$

So

$$\text{Amount in \$} = £300 \times \frac{\$2.10}{£1} = \frac{£300}{1} \times \frac{\$2.10}{£1}$$

And

\text{Amount in \$} = 300 \times \$2.10=\$630.00

Notice that if, given the same rate, you wanted to find the number of pounds you could buy with, say $500, you could reverse the ratio:

$$\text{Amount in } £ = \$500 \times \frac{\$1}{\$2.10} £ = £238.10$$

Key Takeaway

Keep track of the units. You can use them as a guide for calculations.

When dealing with rates, it is critical to keep track of the units; they can guide your calculations if you use them for arithmetic operations.

If, for example, you want Canadian dollars (CAD) for US dollars (USD), and if you know that the exchange rate is 0.68 USD per CAD, then your rates are

$$\frac{0.68 \text{ USD}}{1 \text{ CAD}} = 0.68 \text{ USD per CAD}$$

And

$$\frac{1 \text{ CAD}}{0.68 \text{ USD}} = 1.47059 \text{ CAD per USD}$$

Check this by using a simple example – say, $1,000 (Canadian). Then, by the first rate, you would have:

$$1,000 \text{ CAD} \times \frac{0.68 \text{ USD}}{1 \text{ CAD}} = 680 \text{ USD}$$

By the second rate, you would find:

$$1,000\text{ CAD} \times \frac{1\text{USD}}{1.47059\text{ CAD}} = 680\text{ USD}$$

So the rates are *equivalent*.

The following exchange rates were taken from a local newspaper.

Currency	**Rate**
Euros €	1.532
Hong Kong (HKD)	0.1678
Japan (¥)	0.011648

Each rate gives the number of Canadian dollars required to purchase one unit of the other currency.

Example 1.5.2

To purchase Euros you would use the rate 1.532 CAD per Euro. So, if you wanted 300€ at this rate, you would need:

$$300€ = 300€ \times \frac{1.532\text{ CAD}}{1€} = 459.60\text{ CAD}$$

To get the number of marks for, say $100.00 Canadian, you would calculate:

$$100\text{ CAD} = 100\text{ CAD} \times \frac{1€}{1.532\text{ CAD}} = 65.27€$$

Knowledge Check 3.5

1. Find the Canadian dollars you require to purchase 200 units of each currency listed above.

2. Find the amount of each currency you can buy for $500 Canadian

Your Own Notes

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1.6 Trade Discounts

Most of the goods you purchase as a consumer are not bought directly from the manufacturer. The goods are distributed through a marketing channel – that is, a sequence of middlemen which move the goods to the consumer. In this section, you will examine the price calculations for a typical channel. Consider an industry (such as clothing) which distributes goods through wholesalers and retailers. Note that the manufacturer may sell to different levels of the channel, perhaps to retailers near the production facilities and to wholesalers everywhere else.

Manufacturers (and wholesalers) often produce a single price list of their products for their customers. This usually lists the prices (the *list price*) at the level a final consumer would pay. Clearly wholesalers and retailers cannot pay this price because they have to mark up the price they pay before they sell the product.

The price to be paid by these organizations is found by reducing the list price by a *trade discount*. The trade discount is given by a percent of list. Suppose, for example, that a manufacturer of men's suits lists a suit at $300 (this is roughly what a consumer might pay). Then a retailer buying directly from the manufacturer might get a 25% trade discount and pay:

$$\$300 - 0.25 \times \$300 = \$300(1 - 0.25) = \$225$$

CHAINED DISCOUNTS

If retailers are expected to pay $225 for a suit, a wholesaler has to pay less for it, because the wholesaler's customers are retailers. Hence the wholesaler must get an additional discount to stay in business. This is usually given as a *chained discount;* which is a percentage of the price after the first (retailer's) discount is taken.

In the example above, this extra discount is 20%. So the wholesaler would pay the manufacturer:

$$\$225 - 0.20 \times \$225 = \$225(1 - 0.20) = \$180$$

As a single equation you would have 300(1− 0.25)(1− 0.20). This discount would be described to the wholesaler as list less 20%, 25%.

To get the final price (the net price), after all discounts, you can set up the following formula:

$$N = L \times (1 - d_1) \times (1 - d_2)$$

where:

- N = net price to be paid
- L = list price
- d_1= discount rate 1;
- d_2= discount rate 2

Note that (1-d_1) and (1-d_2) are the fractions of the price to be paid after the discounts are taken. Thus, in the example above:

- For d_1, there remains 1 – 0.25 = 0.75 = 75% to be paid.
- And for d_2 = 20% = 0.20, there is 1 – 0.20 = 0.80 = 80% of the remainder to be paid.

Sometimes there are additional discounts – special sale discounts, seasonal discounts, quantity discounts – that are treated the same way.

Check Your Knowledge 1.6

1. A wholesaler is purchasing refrigerators from a manufacturer and is offered discounts of “list less 30%, 25%” on refrigerators with a list price of $700.00. What price would the wholesaler pay?
2. A retailer purchased a television set from a wholesaler and paid $244.30 after getting a single discount on a set listed at $349. What single discount did the retailer get?

Your Own Notes

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1.7 Cash Discounts

In the previous section you learned to calculate the prices businesses paid for goods. This section examines the *terms of payment* allowed by sellers.

After goods are sold, the vendor sends a request for payment to the purchaser. This request is usually on an *invoice,* a document which lists all the pertinent facts about the transaction, including the following:

- the items sold, with list prices
- the trade discounts
- conditions of delivery
- other charges (e.g., freight)
- the PAYMENT TERMS.

PAYMENT TERMS

Usually payment is not required immediately and a payment period (for example, 60 days) is given. This payment period normally begins on the date of the invoice (that is, the next day is day 1) and payment is expected by the last day of the period. The payment period allows for goods to be partly or wholly resold by the purchasing company before payment is due, and is thus an inducement to buy from the seller.

Key Takeaway

A cash discount encourages early payment.

Sellers have to finance the sale during the payment period and consequently some sellers encourage payment before the end of the period by allowing a discount, called a *cash discount*, for early payment. The discount also helps to ensure preferential payment to the seller by companies that may be experiencing difficulties meeting all payment commitments.

The following are examples of terms:

2/10, n/30	2% discount allowed if paid within 10 days, otherwise net (full) payment due within 30 days (of the invoice date).
1/15, n/60 ROG	1% discount if paid within 15 days of the RECEIPT OF GOODS, otherwise net payment due 60 days from receipt of goods.
2/5, n/30, EOM	2% discount if paid within 5 days of the END OF THE INVOICE MONTH, otherwise full payment due 30 days from the end of the current month.

Key Takeaway

The payment period normally begins on the date of the invoice.

Example 1.7.1

Consider the following invoice:

Ajax Wholesale Ltd.

Invoice No: 16274

Date: August 15, 2020

Ship via: Hermes Delivery

Special Instructions:

Sold to: Electronicks Ltd. None

110 B Avenue Surrey, BC

Ship to: Same PST Exemption **#XXXXXXX**

Terms: 2/5, n/30 FOB

Quantity	Description	List Price per Unit	Amount
15	#1472 AM-FM Radio; less 30%	$17.50	$183.75
10	#1821 Telephone; less 30%	12.25	85.75
10	#2171 Answering machine; less 30%, 10%	36.00	226.80
	Subtotal	496.30	

Delivery charge	35.00
TOTAL PAYABLE	$531.30

Note that the amounts calculated have the trade discounts deducted, but do not have the cash discounts deducted since it is not known whether or not the purchaser will pay early.

The delivery charge is listed separately and cash discounts are not usually allowed on it. This is because the delivery in this case is not the seller's business; it is being paid on behalf of the purchaser.

If the invoice was paid in full by August 20, 2020, the payment would be:

$$\$496.30 \times (1 - 0.02) + \$35 = \$531.37$$

Otherwise the full amount would be due on September 14, 2020.

Knowledge Check 1.7

An invoice dated September 3, 2020, for six television sets with a list price of $749.00 has trade discounts of 30% and 5%. If the payment terms are 3/10, n/30, what is the payment required if the cash discount is taken?

PARTIAL PAYMENTS

Sometimes it is not possible to pay an entire invoice at once. In this case, we can apply the discount to only part of an amount owing. This is called a **partial payment**.

Example 1.7.2

S&C sports receives an invoice for $4,500 for a shipment of curling shoes. The invoice is dated April 1 and has terms 3/10, 2/15, n/30.

a. On April 6, S&C sent in a payment of $1000. The Curling Equipment company gives discounts for partial payments. How much does S&C owe on the invoice?
b. S&C plans to send in a second and final payment on April 11. How much should this payment be?

For part (a), we need to see how much credit should be given for the $1,000 payment, given that it is eligible for a 3% discount. The discount applies to the **credit given** (the part of the invoice that is paid off) and **not** to the **amount paid.**

- To calculate how much to pay **multiply** by (1 – d)
- If given the amount paid **divide** by (1 – d) to get the credit

To see this, we can solve the equation:

$$Credit(1-0.03) = \$1,000$$

Dividing both sides by (1-0.03), we have:

$$Credit = \frac{\$1,000}{1-0.03} = \$1,030.93$$

So, by paying the $1,000 early, S&C now owes only:

$$\$4,500 - \$1,030.93 = \$3,469.07$$

To solve (b), we **multiply** the amount owing by (1-0.02) to see that the final payment is:

$$\$3,469.07(1-0.02) = \$3,399.69$$

Key Takeaways

- Not all suppliers allow retailers to receive a cash discount when only part of the invoice is paid early (partial payments).
- In this text, we always assume retailers will receive a discount on the portion of

the invoice that was paid early.

- You will notice that we did some intermediate rounding in the problem above. Often companies will do this type of intermediate rounding when storing amounts, but it is always important to check with the supplier, to see how they record the numbers.

Knowledge Check 1.8

In the middle of a kitchen remodel, you decide to splurge on a new refrigerator and range. The range was on back order, but you received the following invoice for the fridge:

- Invoice 1 (fridge): $2,500 dated April 4 (5/20, n/30)

On April 6, you sent in a payment of $760. Two weeks later, you receive the invoice for the range. It has the same terms as the first invoice:

- Invoice 2 (range): $3,250 dated April 19 (5/20, n/30)

a. If you pay off both debts at the same time, what is the last day you can pay?
b. How much will you pay?

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
- These notes are for you only (they will not be stored anywhere)
- Make sure to download them at the end to use as a reference

An interactive H5P element has been excluded from this version of the

text. You can view it online here:
https://pressbooks.bccampus.ca/businessmathematics/?p=135#h5p-1

Chap 1 Knowledge Check Answer Key

Knowledge Check 1.1

1. a. \$9,000 b. \$66,000
2. Sounders: $\frac{\$936,400}{\$184,000}$, Elecktronicks: $\frac{\$1,245,900}{\$209,400}$ (Elecktronicks is better)

Knowledge Check 1.2

1. a. Net Profit = 14% of $\$29,000 = 0.14 \times \$29,000 = \$4,060$.

b. $Sales \times 0.14 = \$5,000$, so $Sales = \frac{\$5,000}{0.14} = \$35,714.29$

2 a. \$9.80 b. \$22.40 c. \$39.20

Knowledge Check 1.3

1.a. COGS = 0.65 × \$ 90,000 = \$58,500

Sales	\$90,000
COGS	58,500
Gross Profit	31,500
Operating Expenses	14,000
Net Profit	17,500

2. $\text{Percent gross margin} = \frac{\$31,500}{\$90,000} = 0.35 = 35$

Knowledge Check 1.4

	COST	% MARKUP	MARKUP	PRICE
a.	$10.00	35%	$3.50	$13.50
b.	$20.00	45%	$9.00	$29.00
C.	$16.00	25%	$4.00	$20.00

Knowledge Check 1.5

1. a.

$$200€ \qquad =200€ \times \frac{\$1.532}{1€} \qquad =\$306.40$$

$$200 \text{ HKD} \qquad =200 \text{ HKD} \times \frac{\$0.1678}{1 \text{ HKD}} \qquad =\$33.56$$

$$200¥ \qquad =200 ¥ \times \frac{\$0.011648}{1¥} \qquad =\$2.33$$

b.

$$\$500 = \qquad \$500 \times \frac{1€}{\$1.532} = \qquad 326.37€$$

$$\$500 = \qquad \$500 \times \frac{1\ HKD}{\$0.1678} = \qquad 2,979.74\ HKD$$

$$\$500 = \qquad \$500 \times \frac{1¥}{\$0.011648} = \qquad 42,925.82¥$$

Knowledge Check 1.6

1. $700(1-0.3)(1-0.25) = \$367.50$
2. Discount = $349 - \$244.30 = \$104.70$

$$\text{Discount Rate} = \frac{\$104.70}{\$349} = 0.30 = 30$$

Knowledge Check 1.7

Before Discount: $6 \times \$749(1-0.3)(1-0.05) = \$2,988.51$

After Discount: $2,988.51(1-0.03) = \$2,898.85$

Knowledge Check 1.8

a. The last day you can pay off both debts is the day the fridge invoice is due: May 4. On this day, the fridge will not get a discount, but the range will be eligible for a 5% discount

b. The first payment of $760 is worth a credit of:

$$Credit = \frac{\$760}{(1-0.05)} = \$800$$

so the we owe $\$2,500 - \$800 = \$1,700$ on the fridge, and $\$3,250(1-0.05) = \$3,087.50$ on the range, so our final payment will be:

$$Payment = \$1,700 + \$3,087.50 = \$4,787.50$$

Chapter 1 Review Questions

*Unless otherwise stated, round the **final** answer to 2 decimal places.*

1 A jewelry store's sales and expenses are given below for the months of October, November and December:

	October	*November*	*December*
Sales	$40,000	$59,000	$109,000
Expenses (all)	$38,000	$50,000	$84,000

a. Find the profits for each
b. Find the percent net margin for each

2 FST Computer Stores has decided to give a total bonus of $10,000 to its outlets. The bonus will be proportional to the monthly sales (given below) for each outlet. Find the bonus for each outlet.

Outlet	**Burnaby**	**Downtown**	**Richmond**	**Surrey**
Sales	$162,000	$119,000	$172,000	$215,000

3 A-Plus Appliances two branches reported the following results for a one week period:

	East	**West**
Sales	$20,000	$25,000
COGS	12,000	16,000
Other Expenses	3,000	4,000

a. Find the gross and net profits for each branch.
b. Find the percent gross margin and percent net margin for each

4 Joan’s Co. knows that, nationwide, companies in its industry earn, on average, a percent gross margin of 38% and a percent net margin of 11%. Joan’s Co. forecasts sales of $730,000 next year. If it aims at the same profit ratio as the nationwide average, estimate Joan’s cost of goods, gross profit and net profit.

5 Jim’s Co. sees that its competitor sells an article for $42.00. It knows that the competitor pays $30.00 for the article.

a. What is the competitor’s markup (dollar amount)?
b. What is the competitor’s percent markup?
c. What is the competitor’s percent gross margin?

6 At one time the exchange rate for the Canadian dollar was: 1 CAD = 0.76 USD

For the Danish krone the rate was: 1 krone= 0.155 (USD)

a. Find the value of $1 USD in terms of Canadian dollars (4 decimal places).
b. Find the value of one Danish krone in terms of Canadian dollars (6 decimal places).
c. How much would it cost (in Canadian dollars) to purchase 10,000 krones (2 decimal places)?
d. How many krones can be purchased for $500 Canadian (2 decimal places)?

7 JB Appliances received an invoice which contained the following information:

- Date, June 17, 2018 Items:
- 10 Refrigerators; list $710 each, less 25%, 20%
- 5 Dryers; list $420 each, less 25%, 20%, 5% (special sale)
- Freight charge $208
- Payment terms 3/10 net 60

Find:

a. The last day for
b. The amount due if the invoice was paid on the last day.
c. The last day at which the cash discount
d. The amount of discount allowed on the day in (c).

a. The amount to pay off the invoice on the day in (c)

[8] Mac's Wholesale buys from a distributor which allows discounts of 25%, 20% and 5%.

a. What single discount (called the *single equivalent discount)* would be equivalent to the above discounts?
b. Compare the above discounts to discounts of 30%, 20%.

[9] A branch of Jack's Hardware has been told by its head office to sell goods at a percent gross margin of 40%. What percent markup should the branch use?

[10] CHMCO sells a product with discounts of 25%, 15% to its distributors. It plans to allow a seasonal discount to make the overall discount equal to 45%. What additional (chained) discount should CHMCO allow?

[11] Wilson Co. has received an invoice dated October 17 for 5 items with list price $900 each, and for freight of $170. The terms were:

- Trade terms: list less 20%,15%.
- Payment terms: 3/10, n/30.

Find:

a. The last day for payment
b. The amount due if payment is made on the last day
c. The last day for taking a cash discount
d. The amount needed to pay the invoice on the day in (c).
e. If the seller agrees to give a cash discount on a partial payment, how much credit would be applied to the account if $1,500 were paid immediately?

[12] Three companies, Alpha, Beta and Gamma, are the only suppliers to a specialized market. Last year Alpha's sales were $700,000, Beta's $1,200,000 and Gamma's $1,600,000.

a. What was the market share (percent of market) of each company? ***Note:*** retain all decimal places calculated in (a) to solve (b) and (c).

b. If next year's total market is expected to be $5,000,000 and each company's share remains the same, what would be the sales of each company?

c. Alpha has an aim of making its sales 75% of Beta's. With total sales as in (b) and Gamma's as in (b) what would have to be the sales of Alpha and Beta if Alpha were to make its aim?

[13] Janet's Co. has sales of $90,000, COGS of 80% of sales and operating expenses of $5,000.

a. Find the gross and net profits.
b. Find the rate of markup (based on cost).
c. Find the percent net margin.

[14] Jim's Co. has set a requirement on stock items of a turnover ratio of 2.6 per year. It is examining three stocked items, A, B and C, which have to be bought in large amounts. As a result of the purchasing requirements, the maximum stock for A is $1,000, for B $1,200 and for C $2,500. If the average stock is assumed to be one-half the maximum stock, what would be the required annual sales of each of these items?

[15] Majjor Oil Co. allocates its maintenance budget to three of its branch offices on the basis (i.e., proportional to) of total floor space. Branch I has 9,000 sq. ft., Branch Il has 20,000 sq. ft. and Branch III has 15,000 sq. ft. If $880,000 is to be allocated, how much should each branch receive?

[16] A paint is advertised for sale with a price of $11.00 USD per gallon. Find the price in CAD/liter if 1 CAD = 0.73 USD

Given:

- 1 liter = 1.0567 quarts
- 1 gallon = 4 quarts

[17] The wellhead price of a certain grade of oil is $69.00 USD per barrel. What is the price in CAD per liter?

- 1 CAD = 0.73 USD

- 1 Barrel = 31.5 US Gallons
- 1 liter = 1.0567 quarts
- 1 US Gallon = 4 quarts

18 An invoice arrives from the Home Shopping Network. The invoice is dated May 28th, and the invoice amount is $600. Calculate the net payment of the invoice if the te1ms are 2/10, n/30 and the invoice is paid on June 6.

19 An invoice dated Aug. 12th for $745 containing the terms “3/15, n/60” was paid on Aug. 25th How much should the cheque be to pay this invoice?

20 An invoice dated May 4th for $1,200 has terms 3/10, 2/30, n/60. How much should be paid on May 31st to fully pay off the invoice?

21 An invoice dated May 1st for $4,000 has terms 3/10, 2/20, n/45. It was partially paid by a $1,000 payment on May 10th and a second payment of $1,500 on May 20th. What is the outstanding balance after the May 20th payment?

22 A dealer purchased 10 DVD players at $300 each less discounts of 25% and 5%, and 5 TV’s at $450 each less 30%, 10% and 2%. The invoice for these items was dated June 5th and had terms 2/10, n/30.

a. If the dealer wishes to take advantage of the cash discount, what would be the last day for payment?
b. What amount must be paid on the day found in part a) to fully pay it off?
c. If the dealer only sent in $2,500 on June 15th what would be the outstanding balance?

23 An invoice for $12,000 has payment terms 3/10, 2/20, n/45. Discounts are allowed for partial payments. The company made of payment of $4,000 9 days after the date of invoice, and a second payment 18 days after the date of the invoice that reduced the balance owing to $2,000. What is size of the second payment?

24 An invoice for $2,500 has terms 2/10, n/30. The invoice is dated April 27th and half of the invoice amount is paid on May 1st and the rest is paid on May 20th. If the seller grants discounts for partial payments, by how much is the balance reduced by the 1st payment? What is the total amount paid?

25 A seller grants discounts for partial payments. An invoice for $1,200, dated May 1st, has terms 2/10, 1/15, n/60 and $400 is paid on May 5th, $400 on May 14th and the rest is paid on June 5th• How much is the balance reduced by the two early payments? How much is paid on June 5th?

26 Chan Enterprises sent CK Air an invoice dated Sept. 14th for $5,400 with terms 3/10, 1.5/20, n/45. CK Air made a payment of $2,000 on Sept 24th and a second payment on Oct 3 that reduced the balance owing to $1,000. What was the size of the second payment?

27 You receive an invoice from AG Electronics dated Feb. 10th for:

- 20 TV sets at $500 per set less 20% and 15%.
- 30 DVD players at $200 each less 25% and 10%.

AG Electronics has payment terms 2/10, 1/20, n/45. Cash discounts are allowed for partial payments. They make of payment of $4,900 on Feb 18th and a second payment on Feb 26th that reduced the balance owing to $1,500. What is the size of the second payment?

28. The Harrison Lake Corporation likes to maintain a capital structure of 5:4:2 (dollar value of debt to common shares to preferred shares). The company is considering a new project and will need to raise $3,300,000. If the company wishes to maintain its existing capital structure, how much debt, common stock and preferred stock should they issue?

29 The Ministry of Education allocates its budget in proportion to student enrollments. The Ministry has $1,500,000 to allocate and the student enrollments are as follows:

High Knowledge Institute, The	12,000
Institute of Technology	15,000
School of Hard Knox	13,000

a. Allocate the budget in proportion to
b. The following year total enrollment is expected to reach 65,000. High Knowledge

Institute is expected to have 3,000 more students and the ratio of students at The Institute of Technology to Students at Hard Knocks will remain the same at 15:13. Give the new estimate for enrollment at each

30 An engineering firm is owned by three partners (Alan, Barbara, and Chuck) with 4,000, 3,000 and 1,000 shares respectively. The company's net income for the year is $4,500,000 and 10% of that income has been set aside as a performance bonus. The bonus will be allocated in the following way:

- $70,000 to be distributed amongst all staff excluding the partners.
- Each partner will get $80,000.
- The remainder is to be divided among the partners in proportion to the number of shares held.

What was the total amount of bonus received by each partner?

31 The $300,000 yearly rent of a four-story building is to be expensed to the four departments using it according to the number of floors occupied. Sales occupies two floors, Finance one floor, Administration and Research and Development (R & D) share one floor equally. How much of the rent expense is allocated to each department?

32 Last year, Reliable Securities established a sales achievement bonus fund of $10,000 to be distributed at the year's end among its four-person mutual fund sales force. The distribution is to be made in the same proportion as the amounts by which each person's sales exceed the basic quota of $500,000. How much bonus will each salesperson receive from the fund if the sales figures for the year were $910,000 for Alicia; $760,000 for Bob; $460,000 for Charles; $630,000 for Diana?

33 For the past seven years the sales of Departments A, B, and C have maintained a relatively stable ratio of 4:3:2. Department A is predicting that their sales will be $500,000 next year. Based on their past sales ratio, what sales would be predicted for Department B?

34 If the chain discount on an item listed at $2,200 is 40%, 15%, 5%, what is the net price? What single rate of discount is equivalent to this series of discounts?

[35] The Ray department store is selling all summer clothes at 50% off the regular retail price during the month of September. On Saturdays only, they are offering an additional 20% off the already discounted price. What is the single equivalent discount rate on Saturdays?

[36] A publisher sells its romance novels with chained discounts of 20% and 10%. The publisher would like to add a third discount to bring the overall discounts to 31.6%. Find the third discount rate.

[37] If the net price of a typewriter was $425.59 after chain discounts of 25%, 10% and 3%, what was the list price? What is the equivalent single discount rate?

[38] A patio set is listed at $350. What is the size of the discount (in dollars) if the buyer is eligible for discounts of 30%, 15% and 5%? Convert the chain discount into a single equivalent discount rate.

[39] Which offers the larger discount? Manufacturer A offering chain discounts of 30%, 20%, 5%, 2% or Manufacturer B giving 45%, 13%?

[40] Island Remanufactured Wood Products Inc. sells its products at trade discounts of 30%, 10%. A competitor has been offering products at the same list prices but with trade discounts of 25%, 20%. Island Reman wishes to beat the competitor's prices by offering a third trade discount. At least how big must the additional discount rate be to meet this objective?

[41] Toys-R-Costly sells Barbie Computers for $840 less 20% and 15%. A competitor sells a similar computer for $800 less 25%. What further discount (second discount) must the competitor allow so that its net price is the same as Toys-R-Costly's?

[42] In order to stimulate sales of file cabinets, a manufacturer had to offer a temporary additional discount to its present rate of 25%. The list price is $300. What additional temporary discount would have to be given in order to achieve a net price of $150?

[43] Wadi Air Conditioners Ltd. sells one model at a list price of $500. Retailers are offered discounts of 15% and 10%.

a. Find the net price of an air conditioner.
b. Find the single equivalent discount rate
c. Wadi would like to add a third chained discount to bring the total discount to 27%. Calculate the rate of the third discount.

[44] A company manufactures Bluetooth Speakers. The Speakers retails for $400 and are offered to the wholesaler with chained discounts of 10% and 15%.

a. Find the net price after discount.
b. What is the single equivalent discount rate?
c. The manufacturer would like to add a third discount to bring the total discount to 30%. Calculate the rate of the third discount.
d. A competitor offers a single discount of 25%. This amounts to a discount of $87.50 off the list price. What is the list price?

[45] After graduation you decide to start up a business selling spices imported from Colombia. The main product you import, saffron, sells for 95,149.87 Pesos for a 4-ounce jar. In Canada, you sell the product in 125 gram bags. Find the selling price in Canadian dollars per 125 gram bag. Use the following:

- 1 pound = 454 grams
- 1 USD = 1.5644 CAD
- 125 grams = 1 bag
- 16 ounces = 1 pound
- 0.000427 USD = 1 COP (Columbian Peso)

[46] You are taking a holiday in Britain. You are taking $750 USD with you. How many British pounds can you buy with this money?

Rates:

- $1 USD= $1.5644 CAD
- 1 British pound (£) = $2.1356 CAD

[47] You are thinking of expanding your bottled water company to the US. Your company bottles pure Fraser River water collected near the Oak Street Bridge. Your water currently sells for $0.85 per 600 ml bottle. The water shipped to the US would be sold in quart bottles. What price, in US dollars, should you charge for a quart of your water? (Convert the Canadian price per 600 ml to a US price per quart).

Rates:

- $1 USD= $1.5644 CAD
- 1 Gallon = 3.7853 Litres
- 4 Quarts = 1 Gallon
- 1,000 ml = 1 Litre

[48] You missed the Back Yard Boys concert at GM Place, and have decided to drive to Seattle to see them. After driving through customs, you stop at the Thrifty market in Blaine. You see a quart size bottle of Duff Cola selling for $1.89 USD. You would like to compare this price to the price of a Canadian can of Cola. Convert the price to Canadian dollars per can.

Rates:

- $1 USD = $1.5644 CAD
- 1.0567 Quarts = 1 Litre
- 1 Litre = 1,000 ml
- 1 can = 355 ml

[49] In California gasoline sells for $1.99 USD per gallon. In British Columbia gasoline is currently selling for $0.729 CAD per liter. Convert the Canadian price per liter to US dollars per gallon. Is the price of gasoline cheaper in California or in BC?

Rates:

- $1 USD= $1.5644 CAD
- 1 Litre = 1.0567 Quarts
- 4 Quarts = 1 Gallon

[50] You purchase gasoline for your gas stations from local refineries. A number of labour

disruptions and increasing costs have motivated you to consider importing gasoline from the Emirate of Abu Dhabi. Refined gasoline sells for 91.30 Dirhams per barrel. Calculate the equivalent Canadian price in liters given:

Rates:

- 3 Dirhams = 1 USD
- 0.638 USD = 1 CAD
- 1 Barrel = 42 Gallons
- 1 Gallon = 4 Quarts
- 1 Quart = 0.9464 Liters

[51] Fill in the Following Table:

Cost	**Sales Price**	**Markup in dollars**	**% Markup**	**% Margin**
$125	$175			
$75	$95			

[52] A computer dealer buys NADIR brand VR headsets for $3,500 less 10%, 5% and sells them for $3,299.

a. What is the Cost?
b. What is the rate of markup?
c. What is the percent margin?

[53] An outboard motor costs the retailer $500 less chain discounts of 30%, 25% and 5%: The company maintained a margin of 40% on all items. The motor was sold after it had been marked down 25%.

a. What was the regular selling price?
b. What was the actual selling price (sale price)?
c. What rate of markup did they use? (based on part b)

[54] A souvenir stand bought 300 hockey sweaters for $15 each. They sold 170 at the regular selling price of $30 each but had to sell another 70 sweaters at a 35% discount (markdown) near the end of the hockey season and the remaining sweaters were cleared by selling them

at a breakeven price which exactly covered the cost of the sweaters plus overhead. Overhead (operating expenses) are roughly 25% of cost.

a. What was the selling price per sweater for the 70 sweaters sold near the end of the season?
b. What was the selling price per sweater for those sweaters sold at the breakeven price?
c. What was the gross profit (loss) and net profit earned from the sale of the 300 sweaters?

55 Wilma Inc. prices its products to provide a 25% margin. What rate of markup do they use?

56 You know that a TV retailer makes $50 on the sale of a certain model of television set and the retailer has a markup policy of 20% of cost.

a. How much did this TV set cost the retailer?
b. What is the selling price of the TV set?
c. Find the percent margin.

57 A diamond ring sells for $4,500. If the rate of markup is 110%, what did the ring cost the retailer? What is the percent margin?

58 An item that cost the dealer $350 less 35% and 12.5% carries a price tag at a markup of 150% of cost. For quick sale, the item was reduced 30%. What was the sale price?

59 The selling price of an automobile is $9,500. If the markup is 15% of cost and the operating expenses are 2% of the selling price, how much is the gross profit (markup in $)? What is the net profit?

60 ABC Co. prices its products to provide a 43.5% margin. What rate of markup do they use?

61 A bookstore has a policy of maintaining a margin of 20%.

a. If an item costs $90, how much should it sell for?

b. If another book sells for $120, how much did it cost?

[62] The regular selling price of merchandise sold in a store includes a margin of 40%. During a sale, an item that costs the store $180 was marked down 20%. For how much was the item sold for?

[63] An item sells for $500. If the company uses a 100% rate of markup, what is the cost of the item?

[64] Scoopy-Doo Pet Foods prices its pet food to maintain a 45% margin. A 7-kg bag of Woof-Woof dog food costs $27.50. Calculate the selling price.

[65] CK Winery sells its merlot red wine for $21.00. If the rate of markup is 200%, what did the wine cost the retailer? What is the percent margin?

[66] If you knew that an appliance dealer makes $100 on the sale of a certain model of washing machine and the dealer has a markup policy of 20% of cost, how much did this washer cost the dealer? What is the selling price of the washer? Find the percent margin.

[67] Find the cost of an item sold for $1,904 to realize a rate of markup of 40%.

[68] A retailer bought an article for $78 and his rate of markup is 32%. He sold this article in a sale after having marked it down by 30%. What is the sale price?

Solutions

Notes

1.

	October	November	December
a.	$2,000	$9,000	$25,000
b.	5.00%	15.25%	22.94%

2.

Burnaby	Downtown	Richmond	Surrey
$2,425.15	$1,781.44	$2,574.85	$3,218.56

3.

		East	West
a.	Gross Profit	$8,000	$9,000
	Net Profit	$5,000	$5,000
b.	Gross Margin	40%	36%
	Net Margin	25%	20%

4.

Sales	$730,000
COGS	452,000
Gross Profit	277,400
Operating Expenses	197,100
Net Profit	80,300

5.
   a. $12,000,
   b. 40%
   c. 28.57%

6.
   a. 3158 CAD = 1.00 USD
   b. 0.203947 CAD = 1 Krone
   c. 10,000 Krones =2,039.47 CAD
   d. 500 CAD = 2,451.61 Krones

7.
   a. August 16, 2018
   b. $5,665.00
   c. June 27, 2018
   d. $163.71

e. $5501.29

8. a. 43%

b. 44%

9. 66.67%
10. 13.7255%
11. a. November 16

b. $3,230.00

c. October 27

d. $3,138.20

e. $1,546.39

12.

	Alpha α	Beta β	Gamma γ
a.	20.00%	34.2857142%	45.7142857%
b.	$1,000,000	$1,714,285.71	$2,285,714.29
c.	$1,163,265.31	$1,551,020.41	$2,285,714.29

13. a. Gross Profit= $18,000, Net Profit= $13,000

b. 25%

c. 14,44%

14. A: $1,300 B: $1,560 C: $3,250
15. Branch I = $180,000; Branch II = $400,000; Branch ID= $300,000
16. $3.98 CAD/liter
17. $0.79 CAD/liter
18. $588
19. $722.65
20. $1,176.00
21. $1,438.46
22. a. June 15th

b. N = $3,456.12

c. $3,526.65 - $2,551.02 = $975.63

23. $5,758.76
24. $1,275.51 is credited and a total of $2,474.49 is paid
25. $408.16 + $404.04 = $812.20 is the credit and $387.80 is still owing.
26. $2,303.07
27. $4,306.50

28. Debt: $1,500,000 C/S: $1,200,000 P/S: $600,000
29.
    a. $450,000; L: $562,500; S: $487,500
    b. N: 15,000 students L: 26,786 students S:23,214 students
30. Alan: $150,000; Barbara: $132,500 Chuck: $97,500
31. Sales: $150,000; Fin: $75,000; Admin: $37,500 (same for R & D)
32. Alicia: $5,125; Bob: $3,250; Charles: 0; Diana: $1,625
33. $375,000
34. N = $1,065.90 D = 51.6%
35. 60%
36. 5%
37. L = $650.00; D = 34.525%
38. $152.16, 43.5%
39. Mfg A: Total discount= 47.864%; Mfg B: Total discount= 52.15% (B is larger)
40. more than 4.76%
41. 4.8%
42. 33.33%
43.
    a. N = $382.50
    b. 23.5%
    c. 4.575%
44.
    a. $306
    b. 23.5%
    c. $d_3 = 8.50\%$
    d. $350
45. $70/bag
46. £549.40
47. 0.856959 USD/quart, so $0.86 USD/quart
48. 1.11 CAD/can
49. 1.76 USD/gallon, so cheaper in Canada
50. 0.30 CAD/Liter

51.

Cost	Sales Price	Markup in dollars	% Markup	% Margin
$125	$175	$50	40%	28.57%
$75	$95	$20	26.67%	21.05%

52.
    a. Cost = $2,992.50
    b. 10.24%

c. 9.29%

53.
a. $415.63
b. $311.72
c. 25%

54.
a. $19.50
b. $18.75
c. $3,090 gross profit, $1,965 net profit

55. 33.33%

56.
a. C = $250
b. S = $300
c. % margin = 16.67%

57. C = $2,142.86, percent margin= 52.4%
58. $348.36
59. Markup = $1,239.13 NP= $1,049.13
60. 76.99%

61.
a. $112.50
b. $96.00

62. $240.00
63. $250
64. $50
65. C = $7.00, % margin= 14/21 = 66.67%
66. C = $500, S = $600, % margin = 16.67%
67. $1,360.00
68. $72.07

Chapter 2: Functions and Applications

Key Takeaways

The purpose of a function is to enable you to find the value of a quantity (variable) you want from a quantity (variable) you know or can choose.

Most quantities in business (for example, profits, costs, sales) can take on a wide range of values (profits can even be negative!). These quantities are referred to as *variables,* and there are relationships between many of these variables that help you to plan and improve the operation of a business. The relationships examined in this chapter are called *functions.*

The *purpose* of a function is to enable you to find the value of a quantity (variable) you *want* from a quantity (variable) you *know* or can *choose.* The value of the quantity you want thus *depends* on the value of the quantity you know. The quantity you start with (the known one) is called the *independent* variable; the quantity you want is the *dependent* variable.

An *independent* variable can take on any value consistent with the situation being investigated. A *dependent* variable must take on the value that is a consequence of the known value of the independent variable.

These variables are often denoted by x for the independent variable and by y for the dependent variable. The letters are just a convenient shorthand, which makes it easy to do algebraic manipulation. In applications you will generally find it preferable to use letters that remind you of what is being represented (for example, cost).

2.1 Example of a Function

Example 2.1.1

A shirt manufacturer may decide to produce a certain number of shirts of a particular size and style. The first task is to find out the amount of material needed.

Then,

$$\begin{array}{lll}\text{Amount of Material} & = m & \rightarrow \text{DEPENDENT VARIABLE } (y) \\ \text{depends on} & & \rightarrow \text{IS A FUNCTION OF} \\ \text{Number of Shirts} & = n & \rightarrow \text{INDEPENDENT VARIABLE } (x)\end{array}$$

A function must be described in such a way that you can find the value of the dependent variable from the value of the independent variable. Three standard ways of doing this are used in this course.

They are,

- an equation, which allows you to calculate the value of the dependent variable for each value of the independent variable.
- a table, which lists the value of the dependent variable for each value of the independent variable.
- a graph, which displays the value of the dependent variable for each value of the independent variable.

Example 2.1.2

Suppose a series of items are to be marked up by 40% of cost. The selling price (the value wanted) can be described in terms of the cost (the value known) in the following ways:

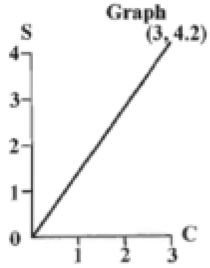
Equation

$S = C + .4C$ or
$S = 1.4C$

Table

C	S
1.00	1.40
1.50	2.10
2.00	2.80
2.50	3.50
3.00	4.20

Graph



All of these approaches give the selling price, (dependent variable), as a function of the cost, (independent variable). A function like this, where one variable is a multiple of another, is called *direct variation.*

Note that such functions follow the algebra convention which states that when single letters are used to stand for variables, multiplication is to be used when no other operation is given, so that

$$1.4C \text{ means } 1.4 \times C$$

Key Takeaways

Equations provide short descriptions of functions and enable you to find precise values of the dependent variable.

Also, you will follow the graphing convention which requires that the independent variable be drawn on the horizontal axis and equal the dependent variable on the vertical axis.

Each way of describing functions has its own advantages. Equations have the advantage of providing short descriptions of functions and enabling you to find precise values of the dependent variable when you need them. However, it may be difficult to do the calculation, or to understand the overall behavior of the function.

Key Takeaways

Tables give values immediately, without calculation … Graphs communicate an overall *understanding of the way a function behaves.*

Your Own Notes

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2.2 Linear Functions

Tables give the values immediately, without calculation, but they may have to be very large to be useful. (Clearly the small table used in Example 2.1.1 is not large enough to be of real use in a business.) Tables are still commonly used in some areas of business – for example, to find the payments required for loans such as mortgages. They are useful because the calculations are fairly complicated and a precise answer is required. Later chapters deal with such payments. Normally we use spreadsheet programs such as Microsoft Excel for this type of problem.

Graphs communicate an overall understanding of the way a function behaves. Complex functions are often referred to by graphs. The function values and equations may not be known in detail, and the general behavior of the function may be all that is required. In economics, for example, the major issue may be the direction (up or down) that changes in one variable (such as price) have on another variable (such as revenue).

The rest of this chapter examines a type of function often used in industry and finance: linear functions.

A function is *linear* if the graph of the function forms a straight line. You will get a straight-line graph if the only operations on the independent variable are as follows:

1. The independent variable is multiplied by a constant. For example, $S = 1.4C$; *Or*
2. The independent variable is added to or subtracted from a constant. For example, $y = x + 4$; *Or*
3. A combination of 1 and 2 is used. For example, $t = 5 + 2n$

All of the above would qualify as linear functions. Note that direct variation is a special case of linear function.

The following equations would result in non-linear functions:

$$y = x^2 - 7 \ \ (a\, parabola)$$

$$v = 4 + \frac{2}{n} \ \ (a\, hyperbola)$$

Example 2.2.1

Take the cost of producing advertising posters as an example of applying a linear function. Suppose that the management of the print shop finds that producing the posters for local companies costs $80.00 for artwork and set-up, and an additional $0.11 for materials and labor for each poster printed.

The cost of production could be described by the (linear) equation

$$C = 80 + 0.11x$$

where

- C is the cost of production and
- x is the number of posters produced

Number Of Posters	**Cost**	
100	$ 91.00	*This table would be satisfactory*
200	102.00	*for costing orders to the printer*
300	113.00	*as long as orders were limited*
400	124.00	*to these sizes.*
500	135.00	
600	146.00	
700	157.00	
800	168.00	
900	179.00	
1,000	190 .00	

To make a graph showing this function you should take as a horizontal axis the independent variable x (number of posters), and as a vertical axis the dependent variable C (cost). Both axes must be scaled so as to allow for the largest value of each variable (2,000 for x , $300 for C). Plotting the points from the table above, you get, by joining them, the graph illustrated below.

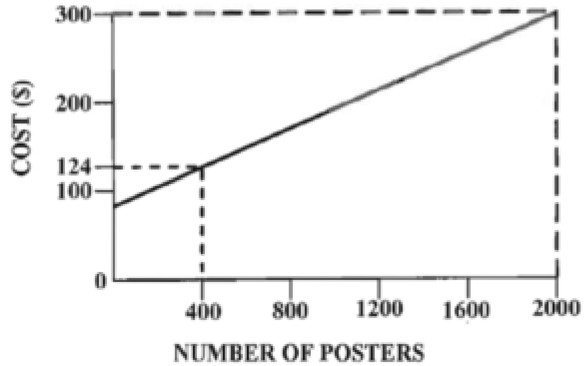


Figure 2.2.1 Graph of Cost (C) of Posters (x)

The values corresponding to a few points have been marked on the graph and dotted lines given to show how these points relate to the axes. Note that the graph is a *straight line.*

All linear functions are quite simple: only two points are necessary to establish the appearance of the function. An extra point serves as a check to see that the graph is correct.

Key Takeaways

Key fact about linear functions is the uniform rate of change, or slope of the line.

Also note that as the independent variable (number of posters) increases, the dependent variable (cost) increases at a steady *rate* ($0.11 per poster).

The key fact about linear functions is this uniform *rate of change,* which is called the *slope* of the line. If you revert to the generally used and for independent and dependent variables.

Then,

$$slope = \frac{rise}{run} = \frac{\text{change in } y}{\text{change in } x}$$

= change in *y* per unit change of *x*

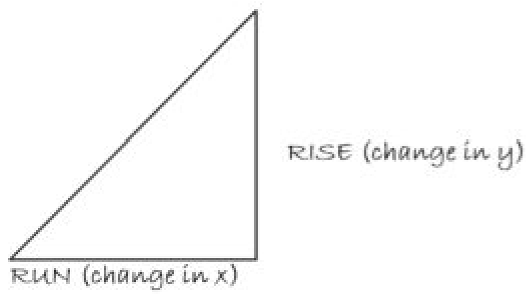


Figure 2.2.2 Rise over Run

The slope will always have the value of the coefficient (multiplier) of if y is written as a function of x.

Thus,

$$y = a + bx$$

Where,

- *a* and *b* are constants.
- *a* is the y-intercept (y-value at = 0).
- *b* is the slope of the line.

Key Takeaway

When *x* changes by 1, *y* will change by an amount *b*, which increases if the slope is positive and decreases if it is negative.

Example 2.2.2

As an example, suppose you have the following linear function:

$$y = 3 + 2x$$

Then the slope will have the value +2. Make a table of a few values and graph the function. Just substitute selected values of *x* to get values of *y*.

For example, if $x = -2$:

$$3 = 2x = 3 + 2(-2) = 3 - 4 = -1$$

x	y
-2	-1
-1	1
0	3
4	11

Key Takeaways

1. When *x* changes by 1, y changes by 2.
2. When *x* changes by 4, *y* changes by 2 × 4 = 8.
3. Also $a = 3$ is the value of *y* when $x = 0$.

The graph would then appear as shown in Figure 2:

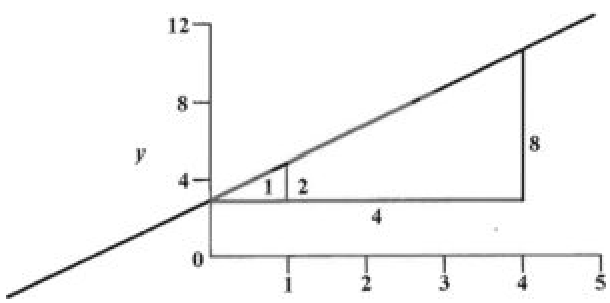


Image 2.2.3: Graph of y = 3+ 2x

Knowledge Check 2.1

1. For each of the following linear equations find the intercept and the slope.

a. $y = 6 + 3x$

b. $t = 2n + 4$

c. $5s = 10 - 5r$

2. A machine that wraps boxes of candy requires 90 minutes to set up, then one-half minute *per box* to do the wrapping. Use

- T = total time required (in minutes)
- *x* = number of boxes to be wrapped

a. Complete the equation for T. Note the units used.

Total time = set up time + wrapping time.

$$T(min) = \_\_\_ \times \_\_\_ \frac{minutes}{box} \times x\ boxes$$

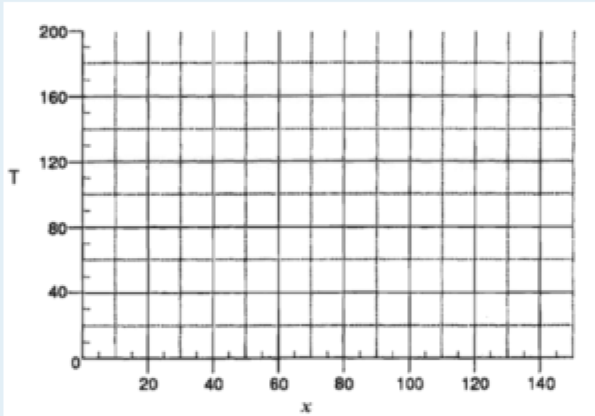
without units:

$$T(min) = \_\_\_ \times \_\_\_ \frac{minutes}{box} \times x\ boxes$$

b. Complete the table for the given x-values:

x	T
0	
80	
150	

c. Plot the points from your table on the graph and join them to form a line.



d. Estimate the time to wrap 100 bags and check by calculating it with the function.

LINEAR EQUATIONS FROM DATA

You will not always be given the equation for a linear function. Instead you will have available some information about it and, from that data, you will have to work out the equation.

The most common case is to have two data points, (x_1, y_1) and (x_2, y_2) which satisfy the equation. Then, since you are working with the slope-intercept form of linear equation, you should first work out the slope, *b*, as follows:

$$Slope = b = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

Next, substitute one data point, say (x_1, y_1) into the equation to find the intercept, a, as follows:

$$y_1 = a + bx_1$$

which can then be solved for the intercept as it is the only unknown.

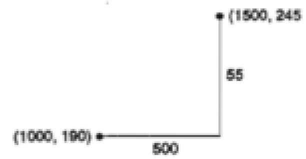
Example 2.2.3

Suppose that in the poster printing example above you were only given the facts that, to make 1,000 posters, it would cost $190 and, to make 1,500 posters, it would cost $245. The information could be organized by thinking of the points as being in a table or on a graph as follows:

Table

p	C ($)
1,000	190
1,500	245

Graph



Then to find the slope:

$$slope = b = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\$245 - \$190}{1,500 - 1000} = \frac{\$55}{500} = \$0.11 \text{ per poster}$$

To find the y-intercept:

$$\$190 = a + \$0.11 \times 1,000$$
$$a = \$190 - \$0.11 \times 1,000 = \$190 - \$110 = \$80$$

and write the whole equation as follows:

$$C = \$80 + \$0.11x$$

Knowledge Check 2.2

There are two temperature scales in North America: Fahrenheit and Celsius. They are related by a linear equation. The temperature of boiling water is 212° Fahrenheit and 100° Celsius. The temperature of freezing water is 32° Fahrenheit and 0° Celsius.

Let,

F = temperature on the Fahrenheit scale (the *y*).
C = temperature on the Celsius scale (the *x*).

Then find the equation for F in terms of C (i.e., $F = a + bC$) by using the following steps:

1. Choose one point as (x_1, y_1) another as (x_2, y_2) in the formula, $slope = b = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{?}{?} = ?$
2. Since you now know b, substitute b and one of the known points in $F = a + bC$

and solve for a.

3. Write the final equation and find the Fahrenheit temperature equivalent to 20 degrees Celsius (approximately room temperature).

Your Own Notes

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2.3 Cost Functions

Key Takeaways

Cost functions give the cost of an activity.

The function used as an example in this chapter, which gives cost in terms of a number of posters, has some elements typical of *cost functions*. Cost functions give the cost of an activity, usually in terms of some measurement of a level of activity.

The ***fixed cost*** is the cost that must be paid no matter how low the measured level of activity (the cost at zero). For the posters the fixed cost was the $80.00 for artwork and set-up, since these things would have to be done even if only one poster was produced.

The ***variable cost*** is the part of the cost that changes with the level of activity. For linear cost functions, the variable part of the cost changes **directly** with the level of the activity. In the poster example, the variable cost is $0.11*x*. The $0.11 per poster is called the *variable cost per unit*. The fact that the variable cost per unit is the same for all units makes the function linear.

Key Takeaways

Variable cost: that part of the cost that changes with the level of the activity

- The fixed cost is usually denoted by F or FC.
- The variable cost per unit is usually denoted by V or VC.

If you take x to be the number of units produced, then you have the following as a general formula:

$$C = FC + VCx$$

Thus, for the poster example:

- FC = $80.00
- VC=$0.11 per poster
- x = number of posters

Hence,

$$C = \$80 + \$0.11x$$

Your Own Notes

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2.4 Equations and Functions

Key Takeaways

In linear equations it is always possible to solve for any variable shown in the equation

An equation always shows a relationship between variables, but the relationship is not necessarily to be viewed as a function with independent and dependent variables. For example, in the equation $4p + 3q = 7$, there is no requirement that one variable be independent and the other dependent.

In linear equations it is always possible to solve for any variable shown in the equation -that is, to rewrite the equation with that variable by itself on one side of the equation. If this is done, it is a convention to write the variable on the left-hand side of the equation and treat it as the dependent variable. Thus, in the example above, you could solve for p as follows:

$$p = \frac{7}{4} - \frac{3q}{4}$$

and treat p as a function of q. Similarly you could solve for q as follows:

$$q = \frac{7}{3} - \frac{4p}{3}$$

and treat as a function of p.

Your Own Notes

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2.5 Systems of Equations

If a number of equations involve the same variables, they are called a **system of equations**.

Example 2.5.1

An agent is to purchase two products, G and H, and send them to the company's warehouse. He has a budget of $70,000 and truck space of 9,000 cubic meters. The product costs and sizes are as follows:

	Cost per Case	Volume per Case
G	$10	3 m^3
H	$20	2 m^3

The question to be answered here is: how much of each product should be purchased if both the budget and space are to be used up?

Key Takeaways

Limitations are called constraints. Each constraint gives an equation.

Limitations such as those for budget and space are called constraints. Each constraint gives an equation. To find the equation from the data given, you should consider the totals for each constraint separately and find the contribution of each variable. In this case; the variables are the amounts of each product to be bought. Let,

- g = the amount (number of cases) of G
- h = the amount (number of cases) of H

Then the budget constraint equation can be obtained by considering:

$$\begin{aligned}\text{Total Spent} &= \text{total spent on G} + \text{total spent on H}\\ &= \text{cost per case of G} \times (\text{amount of G}) + \text{cost per case of H} \times (\text{amount of H})\\ &= \$10 \times g + \$2 \times h\\ \$70,000 &= \$10g + \$20h\end{aligned}$$

Similarly,

$$\begin{aligned}\text{Total Space} &= \text{space used by G} + \text{total space used by H}\\ &= \text{space used by G} \times (\text{amount of G}) + \text{space used by} \times (\text{amount of H})\end{aligned}$$

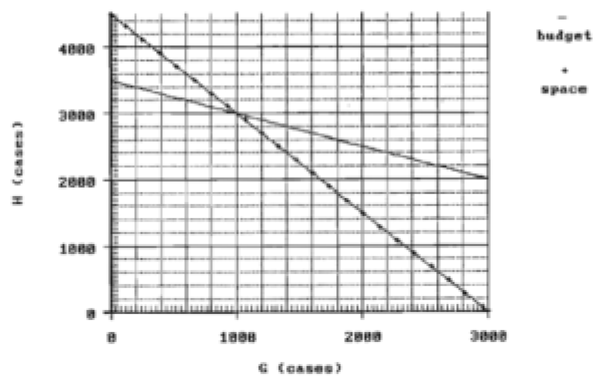
So,

$$9,000m^3 = 3gm^3 + 2hm^3$$

Thus you have the *system of equations*

$$\$70,000 = \$10g + \$20h \text{ and } 9,000 = 3g + 2h$$

for which the graphs are given below.



Graph of G and H, Example 6

Notice that the data for the problem was chiefly in terms of **rates,** the costs **per case** and the volume **per case.** These were used to get equations for totals, using the idea of,

$$Amount = rate \times base$$

for each equation. Notice, also, the use of the units of measurement to help keep track of the parts of the equations.

The values required by the problem are those of the point at which both equations are satisfied – the point on the graph at which the lines cross. This point is called the *solution* of the equations. It can be estimated from the graph and also calculated from the equations.

METHOD OF ELIMINATION

To solve the equations, using the *method of elimination* write each one in the form for which *h* is given as a function of g.

For the budget: $h = 3{,}500 - 0.5g$
For the space: $h = 4{,}500 - 1.5g$

Next, note that at the solution, the values of *h* must be the same, so:

$$3,500 - 0.5g = 4,500 - 1.5g$$

By adding $1.5g$ to each side, you get:

$$3,500 - 0.5g + 1.5g = 4,500 - 1.5g + 1.5g$$
$$3,500 + 1g = 4,500$$

Subtracting $3,500 from each side gives:

$$g = 4,500 - 3,500$$
$$g = 1,000 \text{ cases}$$

Substituting the result in one of the original equations (space):

$$9,000 = 3 \times 1,000 + 2h$$

Whence

$$h = 6,000 + 2 = 3,000 \text{ cases}$$

Thus, both the budget and space allowances will be used up if 1,000 cases of G and 3,000 cases of H are bought.

METHOD OF SUBSTITUTION

The ***method of substitution*** gives you another way of solving such equations. It consists of the following:

Once the first equation has been solved for h, the result is

$$h = 3,500 - 0.5g$$

This result would be substituted in the second equation, which would produce an equation containing only the variable g.

Thus,

$$\begin{aligned}
9,000 &= 3g + 2 \times (3,500 - 0.5g) \\
9,000 &= 3g + 7,000 - g
\end{aligned}$$

and

$$\begin{aligned}
9,000 - 7,000 &= 2g \\
2,000 &= 2g \\
g &= 1,000
\end{aligned}$$

Substituting in the other equation:

$$\begin{aligned}
70,000 &= 10 \times 1,000 + 20h \\
70,000 - 10,000 &= 20h \\
60,000 &= 20h \\
h &= 3,000 \text{ cases}
\end{aligned}$$

Knowledge Check 2.3

One of the production facilities of Jones Furniture Company produces chairs from kits, each of which contains all the parts for the chair. The facility makes two types of chairs: a regular chair for normal inside use and a special chair for outside use. Each chair is assembled and then painted.

There is only a limited amount of labor time for each operation: 120 hours per week for assembly, 80 hours per week for painting. Each regular chair requires 0.5 hours for assembly and 0.3 hours for painting. Each special chair requires 0.4 hours for assembly and 0.4 hours for painting.

Find the number of regular and special chairs that would have to be produced each week in order to use up both the assembly and painting time available for that week.

Let,

r = number of regular chairs produced per week
s = number of special chairs produced per week

1. Find the equation that shows the constraint for the available assembly time: ________ (total hours) = ________ _(hours/chair) × r (chairs) + ________ (hours/chair) × s (chairs)
2. Find the equation that shows the constraint for the available painting time: ? = ? × r + ? × s
3. Solve the equation in Question 1 for r and substitute the result in the second equation, in Question 2, to get the value for I.
4. Substitute the result from Question 3 in one of the equations to get the value of r.
5. Check your results by placing them in the other equation (but not the one in Question 4) and see that they give the correct value.
6. Graph the two equations to make a visual check of your answer.

Your Own Notes

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2.6 Profit Volume Analysis

A system of equations is also used in the analysis of profits for different volumes of business. As an example, consider cost and revenue for the poster example discussed previously. Suppose the printing business decided that it should charge \$0.17 per poster. Then it would have a ***revenue function***:

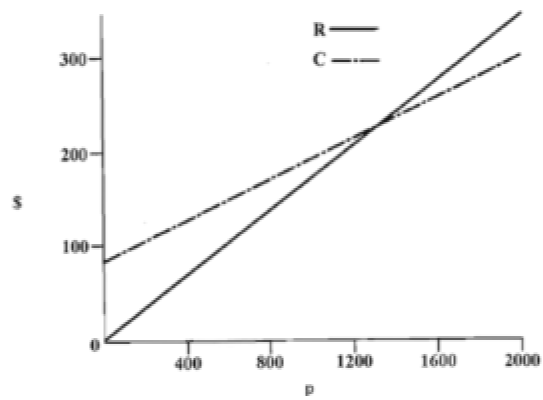
$$R = \$0.17x$$

Together with the cost function, this forms a system of equations:

$$C = \$80 + \$0.11x \text{ and } R = \$0.17x$$

You have already seen a table for the cost function. Now prepare a short table for revenue, then graph both functions on the same axes, as follows:

p	R ($)
0	0
1,000	170
2,000	340



Graph of Cost (C) and Revenue (R)

To analyze this system for business purposes, you should use the relationship (from Chapter 1):

$$Profit = Revenue - Cost$$

You can see from the graph that the printing business would lose money at those values of *x* (the low values) for which the cost function is above the revenue function. And it would make a profit at those values of *x* (the high values) where revenue exceeds the cost.

As in most systems of equations, the point at which all the equations are satisfied – the solution of the equations – is of particular interest. In the example above, it is the place at which,

$$Revenue = Cost$$

That is, the point at which the profit is zero – the break-even point.

The break-even point can be estimated from the graph, but it can also be calculated from the equations in the system.

To find this point, set:

$$Revenue = Cost$$
$$\$0.17x = \$80 + \$0.11x$$

So, by subtracting 0.11*p* from both sides, you get:

$$\$0.17x - \$0.11x = \$80 + \$0.11x - \$0.11x$$

And:

$$\$0.06p = \$80$$
$$\frac{\$0.06x}{\$0.06x} = \frac{\$80}{\$0.06}$$

So

$$x = 1333.33$$

But we cannot make a fraction of a poster! So we always round up.

x = 1,334 posters (rounded up)

This analysis shows the printing company that small orders lose money at $0.17 per poster. In real life such companies cannot accept orders on which they lose money; they will either set a minimum order size (say, 1,500) or charge proportionately more for small orders (for example, $100 for the first 100 posters).

Key Takeaways

- As long as the revenue and cost functions are known, you can evaluate what the profits will be at any level of activity.
- To analyze the relationship between profits and volume for an entire business or product line, do the analysis for a time period.

As long as the revenue and cost functions are known, you can evaluate what the profits will be at any level of activity. Using the revenue function above $R = \$0.17x$,

$$Profit = \$0.17x - (\$80 + \$0.11x) = \$0.06x - \$80$$

so that for, say, 2,000 posters:

$$Profit = \$0.06 \times 2,000 - \$80 = \$40.00$$

The above example was for a **job** in a business. To analyze the relationship between profits and volume for an entire business or product line, do the analysis for a time period, such as a month or a year. As before, the fixed costs are those that do not depend on the **level** of activity. They include:

- rent
- property taxes
- salaries of employees who will be employed regardless of the level of activity (for instance, managers, sales clerks)
- utilities such as heat and light

The variable costs include:

- the costs of materials
- labor costs which are the result of the level of production (called direct labor costs)
- commissions on sales

Example 2.6.1

Suppose a manufacturer of sports bags examined its accounts and found the following costs.

Fixed Costs (Per Month)

Salaries (Office, Sales Management)	$50,000.00
Depreciation of Equipment	8,000.00
Rent	6,000.00
Interest on Loans	4,000.00
Other Fixed Expenses	2,000.00
Total Fixed Cost	$70,000.00

Variable Cost Per Unit Produced

Materials	$4.00
Labour	3.00
Total Variable Cost	$7.00 per bag

Thus the cost equation would be

$$C = \$70,000 + \$7x$$

where

C = cost
x = number of bags produced in a month

This manufacturer sells the bags for $12 per bag. Hence the revenue equation is

$$R = \$12x$$

if we assume all bags produced are sold in the same month as they are produced.

The monthly profit would then be

$$Profit = R - C = \$12x - (\$70,000 + \$7x)$$

$$= \$5x - \$70,000$$

To break even would require:

$$0 = \$5x - \$70,000$$

$$\$5x = \$70,000$$

So

x = 14,000 bags (produced and sold)

Below 14,000 bags the company would show a loss; above 14,000 bags it would show a profit. You can find the level required for any given profit by solving for x with that profit. For example, to earn a $10,000 profit:

$$\$5x - \$70,000 = \$10,000$$

thus

$$x = 16,000 \text{ bags}$$

Knowledge Check 2.4

Suppose, in the above example, new equipment were to be bought, increasing the depreciation cost to $12,000 and the interest on loans to $5,000, but reducing the labor cost to $2.50 per unit. Find the following:

1. The resulting cost function.
2. The volume (number of bags) required to break even.
3. The volume required to make a profit of $10,000.

USING A BAII PLUS CALCULATOR FOR CVP

After graduation you have moved to Australia and discovered a need for bottle openers to take the tops off "stubbies" of pop. You start a company called Pieces of Roo that will make novelty bottle openers shaped like Kangaroo forepaws.

You estimate fixed costs of $2,500 per month and variable costs of $3.50 per bottle opener. The maximum capacity of your factory will be 1,000 units per month. You expect to sell the openers for $12.00 each.

This makes your cost equation:

$$C = \$2,500 + \$3.50x$$

And your revenue equation

$$R = \$12x$$

Use the pre-programmed functions of the BA II plus to answer the following questions:

1. Calculate the breakeven in units.
    - Enter the breakeven worksheet by keying **2nd BRKEVN**
    - Enter fixed costs **2500 ENTER**
    - Scroll down using the down arrow +
    - Enter variable costs **3.50 ENTER**
    - Scroll down using the down arrow +
    - Enter price **12.00 Enter**
    - Scroll down until **Q** appear then press **CPT**
    - The answer is 294.11 which you would round up to 295
2. Calculate the Profit at 1,000 units
    - Change quantity to 1,000 units **1000 ENTER**
    - Scroll down using the down arrow+ until **PFT** is reached
    - **Key CPT**
    - The answer is $6,000
3. You would like to make a profit of $5000 – how many units should you produce?
    - Change PFT to 5000, **5000 ENTER**
    - Scroll down until **Q** appear then press **CPT**
    - 882.35 which you would round up to 883 units

4. At maximum capacity of 1,000 units, you would like to make a profit of $7,500. What price should you charge?
    - Change **Q** to 1,000
    - Change **PFT** to 7500
    - **CPT P**
    - Answer is $13.50 per unit

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
- These notes are for you only (they will not be stored anywhere)
- Make sure to download them at the end to use as a reference

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://pressbooks.bccampus.ca/businessmathematics/?p=209#h5p-1

Chapter 2 Knowledge Check Answer Key

Knowledge Check 2.1

1.	Intercept	Slope
a. $y = 6 + 3x$	6	3
b. $t = 2n + 4$	4	2
c. $5s = 10 - 5r$	2	-1 (note that $s = 2 - r$)

2. a. $T = 90 + 0.5x$

X	T
0	90
80	130
150	165

d. 140 minutes

Knowledge Check 2.2

1. Slope: $\dfrac{212^\circ F - 32^\circ F}{100^\circ C - 0^\circ C} = \dfrac{180^\circ F}{100^\circ C} = 1.8\dfrac{^\circ F}{^\circ C}$
2. Using the point (0°C, 32°F): $\$32 = a + 1.8(0)$, so a = 32°F
3. F = 32 + 1 .8°C. So putting in C = 20, we have $F = 32 + 1.8 \times 20 = 68^\circ F$

Knowledge Check 2.3

1. $120 = 0.5r + 0.4s$
2. $80 = 0.3r + 0.4s$

3. $s = 50$
4. $r = 200$

Knowledge Check 2.4

1. $C = 75,000 + 6.5x$
2. 13,637 bags
3. 15,455 bags

Chapter 2 Review Questions

1 Ocean Marina rents moorage space for boats. Its charge for boats between 20 and 40 feet long is $180 plus $12 a foot.

Let x = boat length in feet C = cost of moorage:

x	*20*	*24*	*28*	*32*	*36*	*40*
C						

a. Complete the above table relating length and cost.
b. State the equation describing cost as a function of length.
c. Graph cost as a function of length for 20 to 40 feet.

2 An agent has available truck shipping capacity for 62.5 tons of materials A and B. Each unit of A weighs 2.4 tons and each unit of B weighs 1.7 tons. Find an equation that relates the amounts of A and B that can be shipped.

3 A company finds that when the temperature is 12°C it uses 100 litres of heating oil per day. When the temperature is 5°C, it uses 170 litres of heating oil per day.

a. Mark the above data points on a graph (allow for negative temperatures).
b. Find the equation (assumed to be linear) for oil usage in terms of temperature.
c. Find the amount of oil that would be used at -5°C. Check your result on a graph.

4 A company has a choice between two copiers, A and B. A costs $120 a month plus $0.05 per page; B costs $250 a month plus $0.03 per page.

a. Find the cost equations for each copier.
b. Which copier would be best for 5,000 pages a month? For 10,000 copies?
c. Graph the equations on the same axes and show clearly where each copier is cheapest.
d. Find the volumes at which the costs are equal and mark the point on your graph.

[5] Jones Stereo Company sells stereo sets for $150 each. The parts for each stereo cost $39.50 and the labor costs $53 per set. Fixed costs are $16,000 a month.

a. Find Jones' cost and revenue functions.
b. How many must Jones produce and sell every month in order to (i) break even? (ii) make a $6,000 profit?

[6] Find the equations of the following lines.

a. Passing through (3, 10) and (I, 6).
b. With slope 1.5 and passing through (2, 5).
c. Passing through (1, 7) and falling 2 units in for each increase of 1 in *x*.

[7] Jimms Company believes that the time required to produce its widgets is a linear function of the number of widgets to be produced in a run. It finds that to produce a run of 600 widgets requires 7,050 minutes, and that to produce a run of 350 widgets requires 4,300 minutes.

a. Find the equation giving time required as a function of the number of widgets.
b. How long will it take to produce 500 widgets?
c. How many widgets can one expect to produce with a run that is assigned 81 hours?

[8] Deluxe Tire Company finds that to produce each additional tire it costs $21.50. The fixed costs of the plant operation amount to $126,000 per month. The tires are sold for $58 each.

a. Find the revenue and cost functions for the operation.
b. How many tires must be produced and sold in order for Deluxe to break even in a month?
c. How many tires must be produced and sold in order for Deluxe to earn a profit of $40,000 in a month?

[9] A company can rent a car using either of two methods of payment. Method A requires $20 per day and $0.20 per kilometer. Method B requires $35 per day and $0.12 per kilometer.

a. Which method is cheaper for 300 kilometres of use in one day?
b. At what level of use per day are the costing methods equally expensive?

c. A competitor to the company charged $37 for a day's use during which 80 km were driven, and $43 for a day during which 120 km were driven. Assume the cost function to be linear and find the cost, C, as a function of distance traveled, *x*.

10 Solve the following system of equations for *x* and *y*:

$$y = 2.0 + 5.2x$$
$$y = 3.0 - 2.0x$$

11 Solve the following system of equations for *x* and *y*:

$$y - 4x = 6$$
$$2x + 3y = 4$$

12 A publishing company finds that, when it prices its computer books at $20 per book, it can sell 7,000 books per month. When it prices them at $25 per book, it can sell 5,500 books per month. The company assumes the relationship is linear.

a. Find the equation which gives the number of books that can be sold per month in terms of the price.
b. How many books should it be able to sell at $22?

13 FG Company produces housings for a major electrical manufacturer. In a four-week period, during which it produced 2,550 housings, its total costs were $120,000, and in a four-week period, during which it produced 1,750 housings, its total costs were $98,000.

Assume that the costs are linear functions of the number of housings produced each four-week period.

a. Find the equation relating total costs to the number of housings produced in a four-week period.
b. According to your equation in (a), what are the following?
    i. the fixed costs per four-week period
    ii. the variable cost per housing
c. If the housings are sold for $55 each, how many must be produced and sold each four-week period in order to break even?
d. Graph your results for costs and revenues. Identify *all* major areas on the graph.

14 A furniture company finds that to make a run of 6,200 cabinets costs $205,000 and run of 8,300 costs $267,000. Assume the relationship between cost and number of cabinets produced is linear.

a. Find the equation which gives cost as a function of number of cabinets produced.
b. If cabinets are sold for $38 each, what is the minimum number of cabinets in a run such that the sales revenue would pay for the cost of the run?

15 For a certain good there is a demand for 8,000 units when the price is $8/unit and a demand for 6,200 when the price is $12/unit. Assuming demand, *d* , is a linear function of price, p, find the slope of the line and the equation.

16 HJ Outdoor Company found that it cost $6,100 to make a run of 105 jackets, and $7,900 to make a run of 150 jackets. Assume the cost is a linear function of the number of jackets made.

a. Find the cost as a function of the number of jackets made in a run, and graph it from 80 to 200 jackets.
b. Estimate the cost of a run of 180 jackets.
c. If HJ sells the jackets for $55 each, how many must be in a run in order to *barely* recover the cost of making the jackets?

17 Find the equation of the line:

a. with slope 3 that contains the point (4, 1).
b. with slope -5 that contains the point (-2, 3).
c. containing the points (2, 3) and (-6, 1).
d. containing the points (12, 16) and (1, 5).
e. containing the points (-4, 5) and (-2, -3).
f. that contains the point (-4, 2) and falls 2 units in *y* for every one unit increase in *x*.

18 You use your calling card to make a telephone call to your friend who lives in Oyster River. You receive your monthly telephone bill and two of the items are as follows:

Time of call	**Area**	**Amount**
5 min	Oyster River	$2.50
11 min	Oyster River	$4.60

a. Define the two variables and create an equation to calculate the total cost of a call.
b. How much would a 60-minute call cost?

[19] You have been asked to predict the cost of flying a Dash 7 aircraft from Vancouver to Seattle. You are provided with the following information:

Number of Passengers	**Cost**
12	$7,680
15	$7,725

a. Define the two variables and create an equation to calculate the total cost of a flight.
b. How much would it cost to fly the plane empty (with no passengers)?
c. Interpret the y intercept using the words of the problem.
d. How much extra does it cost to have one more passenger?
e. Interpret the slope using the words of the problem.
f. How much would it cost to fly 14 passengers?

[20] B. Furniture Co. believes that the cost to produce its chairs is a linear function (straight line relationship) of the number of chairs to be produced in a run. It finds that to produce a run of 130 chairs requires $8,200, and that a run of 250 chairs requires $13,000.

a. Find the equation giving cost, C, based on the number of chairs produced, *x*, in a run.
b. What is the cost if no chairs are produced?
c. Interpret the *y* intercept using the words of the problem.
d. What is the extra cost to make one more chair?
e. Interpret the slope using the words of the problem.
f. How much would it cost to produce 400 chairs?
g. How many chairs can be expected to be produced with a run that is assigned $8,600?

[21] Mr. Smith wants to rent a truck for one day to make a number of deliveries. He can rent the

type of truck he needs from either of two companies. Company A charges $100 plus $0.40 per kilometre. Company B charges $40 plus $0.80 per kilometre.

a. Write equations for the cost, C, of each truck in terms of x, the number of kilometres traveled.
b. Suppose you rent a truck from Company A,
    i. How much would it cost to rent the truck but not actually drive it (0 km)?
    ii. How much would it cost to drive 300 kilometres?
c. Suppose you rent a truck from Company B,
    i. How much would it cost to rent the truck but not actually drive it (0 km)?
    ii. How much would it cost to drive 300 kilometres?
d. Graph the equations on the same axes and show on the graph where each company's deal is best. Graph from 0 to 300 km. Use a large scale, graph paper and straight edge, and apply proper conventions.
e. Using your graph, determine the point where the costs are equal. Mark the point on the graph you made in (d).

[22] Best Buy Furniture Store manufactures and sells bedroom suites. Each suite costs $800 and sells for $1,500. Fixed costs total $150,000.

a. Write down the cost equation and the revenue equation.
b. Find the breakeven point in units (number of suites).

[23] A company has determined that a minimum of 25,000 units of their product can be sold at a selling price of $10 per unit. However, if the selling price was reduced to $8.00 per unit, a minimum of 35,000 units can be sold. Relevant cost data for this product is as follows:

Fixed Costs	**Variable Costs**	
$75,000.00	Labour cost per unit	$4.50
	Material cost per unit	$1.50

a. Determine the breakeven points in units.
b. Determine the profit earned by selling the minimum quantity for each price alternative (i.e., at both $10 and $8).

[24] CK Air has begun a new discount air service to Port Alberni. It costs $2,070 to fly an empty aircraft with crew to Port Alberni. Each passenger costs an extra $12 in food and extra fuel costs. Tickets sell for $150 for a one-way flight.

a. Write down the cost equation and the revenue equation.
b. How many passengers must the plane hold for the company to break even?
c. Calculate the profit if 17 passengers fly to Port Alberni.
d. If the company wants to make a profit of $690 per trip, how many seats must they sell?
e. What is the **contribution margin**?

[25] Finicky-Cat Gourmet Pet Food makes organic cat food and sells them in 2 kg bags. The company has annual fixed costs of $150,000 and variable costs of $4 per bag. Finicky-Cat sells each bag for $10. Production capacity is 50,000 bags per year.

a. Find the revenue and cost equations.
b. How many bags of cat food does the company have to sell to break even?
    i. What are total sales at the breakeven point?
    ii. What is the percent capacity at the breakeven point?
c. The company anticipates that it will be able to make and sell 40,000 bags of cat food this year. What will it cost to produce these 40,000 bags?
d. Graph the cost equation **and** the revenue equation on graph paper. Graph from 0 to 50,000 bags. Make sure to label the axes.
    i. Clearly identify the breakeven point on the graph.
    ii. Identify the area on the graph where the company makes a profit **and** where it has a loss.
e. Finicky-Cat wants to increase its selling price from the current $10 so that it could make a profit of $150,000 from selling 40,000 bags of cat food. What price must they charge?

[26] It costs Brawn Products $8 to make a particular model of shaver. The fixed costs are $12,000 per month. The shavers are sold to retailers for $25 less trade discounts of 33.5%, 10%.

a. Write down the revenue and cost equations.

b. Compute the breakeven point (in units and in sales dollars).

c. Find the profit if 2,200 shavers are sold in a month.

d. Find the profit if monthly sales (in dollars) are $37,500.

e. Brawn wants to make a profit of $5,000/month. How many shavers must be sold?

f. If the fixed cost increased to $18,000/month, how many shavers must they sell to break even?

g. If the fixed costs remain the same (at $12,000/month) but the selling price is reduced to $10, what would the new breakeven point be?

h. Brawn Products wants to add a third discount to reduce their selling price to $10 (as in part g). Find the rate of the third discount.

i. The cost to make a shaver has increased by 25% from the current $8 and fixed costs have fallen by 25% from the current $12,000/month. If Brawn wants to make a profit of $20,000/month from the sale of 5,000 shavers, what should the new selling price be?

[27] Your company needs to lease a photocopier. It has a choice between three photocopiers A, B, and C. Copier A charges a flat fee of $3,000 per month. Copier B charges $2,000 per month plus $0.05 per copy; and copier C charges $1,000 per month plus $0.15 per copy.

a. Find the cost equation for each copier.

b. If the requirement is 15,000 copies per month which copier is cheapest?

c. Calculate the points of indifference based on cost (i.e., determine the number of photocopies which will make the costs equal).

   i. between copier A and B
   ii. between copier A and C.
   iii. between copier B and C.

d. On the same graph, graph the three cost equations over the range 0-30,000 copies.

e. Over what range of values of x (the number of copies) is it most cost-effective (cheapest) to rent from A, B, or C?

Notes

1. a.

x	20	24	28	32	36	40
C	420	468	516	564	612	660

b. [latex]C=180+12x[/latex]

2. 62.5=2.4A+1.7B
3. b. $y = 220 - 10x$, where x is the temperature and y is oil usage; c. 270 litres
4. a. Copier A: C =$120 +$0.05*x* (*x* = number of pages; C = total cost); Copier B: C =$250 +$0.03*x* b. 5,000/month: Total cost for A= $370 Total cost for B = $400 Copier A is cheaper 10,000/ month: Total cost for A= $620 Total cost for B = $550 Copier B is cheaper d. Costs are equal when $x = 6{,}500$
5. a. C =$16,000 +$92.50*x* (C = total costs; R = total revenue; *x* = number of stereos sold); R =$150*x* b. Breakeven = 279 stereos (rounded up from 278.26) For a $6,000 profit number of stereos= 383
6. a. $y= 4 + 2x$ b. $y = 2 + 1.5x$ c. $y = 9- 2x$
7. a. $y=450+11x$ (*y* = time in minutes; *x* = number of widgets) b. 5,950 minutes or 99.17 hours c. 401 widgets
8. a. R= 58*x*; C = $126,000 + 21.50*x*; *x* = number of tires b. 3452.05 round up to 3453 tires c. 4547.9 round up to 4548 tires
9. a. Method B is cheaper by $9.00 b. 187.5 kilometres c. C=$25+$0.15*x*
10. (0.1389, 2.7222)
11. (-1, 2)
12. a. *b* =$13,000 −$300*p*; (*p* = price; *b* = number of books) b. 6,400 books
13. a. C = $49,875 + *$27.50h;* (C = total costs; *h* = number of housings) b. fixed costs= $49,875; variable costs= $27.50 per housing c. 1,814 housings
14. a. C = $21,952 + $ *29.52x;* (C = total cost; *x* = number of cabinets produced) b. 2,589 cabinets
15. $d = 11{,}600 - 450p$
16. a. C = $1,900 + $40*x*; (C = total cost; *x* = number of jackets made) b. $9,100 c. 127 jackets
17.
    a. $y = -11 + 3x$
    b. $y = -7 - 5x$
    c. $y = 2.5 + 0.25x$
    d. $y = 4 + x$
    e. $y = -11 - 4x$
    f. $y = -6 - 2x$
18.
    a. C =$0.75 +$ 0.35*x* where *x* = length of time of the call (in minutes); C = total cost of the call.
    b. $21.75
19.
    a. C = $7,500 + $15*x* where *x* = number of passengers; C = total cost to fly the plane
    b. $7,500
    c. It would cost $7,500 to fly the plane empty (i.e., with no passengers).
    d. $15

e. It costs an extra $15 for each additional passenger.
f. $7,710

20. a. C =$3,000 + 40x$ b. $3,000 c. The cost incurred even when no chairs are produced i.e., the overhead or expenses, (e.g., rent, property taxes). d. $40 e. Each additional chair made will cost $40 (the cost of labour and materials). f. $19,000 g. 140 chairs
21. a. C_A=$100 +$0.40x, C_B=$40+$0.80x b. and c.

# Km x	**Total Cost C_A**	**Total Cost C_B**
0	$100	$40
300	$220	$280

d. Below 150 km it is cheaper to use B; above 150 km it is cheaper to use A. e. The costs are equal at 150 km (both cost $160)

22. a. C = $150,000 + 800x$; R = $1,500$x$; x = the number of bedroom suites produced and sold b. 215 suites (rounded up from 214.3 units)
23. a. 18,750 units/ 37,500 units b. $25,000 profit/ $5,000 loss
24.
    a. C =$2,070 +$12x; R =150x$; x = the number of passengers on the plane
    b. 15 passengers
    c. $276 profit
    d. 20 passengers
    e. $138/passenger
25. a. R =10x$; C =$150,000 +4x$; x = the number of bags produced and sold b. 25,000 bags of cat food, revenue= $250,000; 50% capacity c. $310,000 d.

Number of Bags	**Revenue**	**Cost**
0	0	$150,000
50,000	$500,000	$350,000
25,000	$250,000	$250,000

e. $11.50/bag

26.
    a. R =15x$, C =$12,000 + 8x$, x = the number of shavers produced and sold
    b. 1,715 shavers (rounded up from 1714.28). Sales: $25,725
    c. $3,400
    d. $5,500
    e. 2,429 shavers

f. 2,572 shavers (after rounding up)
g. 6,000 shavers
h. 33.33%
i. $15.80/shaver

27. a. C_A= $3,000; C_B= $2,000 + $0.05*x*; C_C= $1,000 +$0.15*x*; where *x* = the number of copies b. A: $3000; B: $2,750; C: $3,250 so B is cheapest for 15,000 copies c. C_A=C_B at 20,000 copies; C_A= C_C at 13,333 copies; C_B = C_C at 10,000 copies d. Plot these points below. Your lines must cross at the indifference points you found in (c). If not, your graph is wrong.

Number of copies (*x*)	**Cost A**	**Cost B**	**Cost C**
0	$3,000	$2,000	$1,000
30,000	$3,000	$3,500	$5,500

e. A is cheapest for more than 20,000 copies, B is cheapest between 10,000-20,000, copies C is cheapest below 10,000 copies

Chapter 3: Simple Interest

Key Takeaways

Interest is based on the amount of money borrowed, the time allotted for paying it off, and the rate of interest.

If you were to borrow money from an individual or a financial institution such as a Bank or Credit Union, you would expect to be charged a number of dollars for the use of this money. This amount of compensation is called **interest,** and is based on the amount of money borrowed (called the **principal**), the amount of time allotted for paying it off and the rate of interest.

By the same token, if you are to deposit some money in a financial institution, you would expect to get paid interest for allowing the financial institution to use your money. In reality, you are loaning money to the financial institution and they in tum earn an income on this money by either loaning it out or investing it.

The amount of **simple interest** is calculated by using the following relationship:

$$I = P \times r \times T$$

Where:

- I is the amount of interest earned;
- P is the amount of money (principal) borrowed or deposited;
- *r* is the annual rate of simple interest; and
- *t* is the time period in years

Usually, *r* (rate) is quoted as a percent per year (percent per annum or percent pa) and the time (*t*) in years. The units of rate and time **must** match.

3.1 Example of Simple Interest

Example 3.1.1

If the time is 48 days, then

$$t = \frac{48}{365}\ years$$

If the time is 16 months, then

$$t = \frac{16}{12}\ years$$

Note: When the time is given in months, you will treat one month as being equal to $\frac{1}{12}$ year.

If the time is 2 years, then:

$$t = 2\ years$$

Key Takeaways

Be very careful with rounding; be sure to input the exact fraction when using your calculator!

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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3.2 Exact Time

If the two **dates** that determine the interest period are known, it is important to calculate the **exact** number of days between those two dates. To do this, we count the beginning day but not the ending day of the period. We will always use a 365 day year, essentially ignoring leap year.

Example 3.2.1

If the interest period ranges from May 12 to May 27,

$$t = \frac{27 - 12}{365} = \frac{15}{365}\ years$$

Be sure to satisfy yourself that this simple subtraction of dates actually includes the first date but not the last.

You should also know that April, June, September and November have 30 days, February has 28 days (29 days during a leap year) and the remaining months have 31 days. Leap years occur every four years and are determined by getting an integer (a whole number) answer when dividing the last two digits of the year by four.

Example 3.2.2

How many days between April 16 and August 12?

Start Date: April 16 (counted)

End Date: August 12 (not counted)

Number of days:

April	=	(30 – 15)	=	15
May	=		=	31
June	=		=	30
July	=		=	31
August	=	(12-1)	=	11
TOTAL	=		=	118 days

Therefore,

$$t = \frac{118}{365}\ days$$

Some calculators and computer software will calculate the days for the user. In this text we present two methods of date counting: using your calculator, and using a table. We also highly encourage students to take the time to learn how to use the date functionality in Microsoft Excel, as it is well worth the effort.

Using the BA II Plus to Find Exact Dates

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://pressbooks.bccampus.ca/businessmathematics/?p=60#oembed-1

We will use Question 16 from the Practice Problems to illustrate the exact date feature of the BA II plus calculator.

Example 3.2.3

AW borrowed $9,000 on January 30, 2002 and agreed to pay 14% simple interest on the balance outstanding at any time. He paid $5,000 on March 9, 2002 and $2,500 on May 25, 2002. How much did he have to pay on June 30, 2002 in order to pay off the debt?

- First, find the number of days between the debt and the focal date.

2nd Date	DTl =	1.3002 Enter	Enter the date of the loan (month.day year)
	DT2=	6.3002 Enter	Enter the focal date
	DBD=	CPT	**151** from debt to focal date

- Next, find the number of days from the $5,000 payment to the focal date.

2nd Date	DTl =	3.0902 Enter	Enter the date of the first payment
	DT2=	6.3002 Enter	This is the focal date – you do not need to re-enter
	DBD=	CPT	**113** days from the payment to the focal date

- Find the number of days from the second payment to the focal date.

2nd Date	DTl =	5.2502 Enter	Enter the date of the second payment
	DT2=	6.300 Enter	This is the focal date – you do not need to re-enter
	DBD=	CPT	**36** days from the payment to the focal date

We will finish solving this problem later on.

Dates using a Table

If you prefer to use tables, you can use the following:

Day of Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1	1	32	60	91	121	152	182	213	244	274	305	335
2	2	33	61	92	122	153	183	214	245	275	306	336
3	3	34	62	93	123	154	184	215	246	276	307	337
4	4	35	63	94	124	155	185	216	247	277	308	338
5	5	36	64	95	125	156	186	217	248	278	309	339
6	6	37	65	96	126	157	187	218	249	279	310	340
7	7	38	66	97	127	158	188	219	250	280	311	341
8	8	39	67	98	128	159	189	220	251	281	312	342
9	9	40	68	99	129	160	190	221	252	282	313	343
10	10	41	69	100	130	161	191	222	253	283	314	344
11	11	42	70	101	131	162	192	223	254	284	315	345
12	12	43	71	102	132	163	193	224	255	285	316	346
13	13	44	72	103	133	164	194	225	256	286	317	347
14	14	45	73	104	134	165	195	226	257	287	318	348
15	15	46	74	105	135	166	196	227	258	288	319	349
16	16	47	75	106	136	167	197	228	259	289	320	350
17	17	48	76	107	137	168	198	229	260	290	321	351
18	18	49	77	108	138	169	199	230	261	291	322	352
19	19	50	78	109	139	170	200	231	262	292	323	353
20	20	51	79	110	140	171	201	232	263	293	324	354
21	21	52	80	111	141	172	202	233	264	294	325	355
22	22	53	81	112	142	173	203	234	265	295	326	356
23	23	54	82	113	143	174	204	235	266	296	327	357
24	24	55	83	114	144	175	205	236	267	297	328	358

Day of Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
25	25	56	84	115	145	176	206	237	268	298	329	359
26	26	57	85	116	146	177	207	238	269	299	330	360
27	27	58	86	117	147	178	208	239	270	300	331	361
28	28	59	87	118	148	179	209	240	271	301	332	362
29	29		88	119	149	180	210	241	272	302	333	363
30	30		89	120	150	181	211	242	273	303	334	364
31	31		90		151		212	243		304		365

Table 3-1: Days of the Year

Note carefully: For leap years (those years where the last two digits are divisible by four with an integer result), February 29 must be included and becomes day 60. One day must then be added to all dates following. (Examples of leap years: 2016, 2020, 2024, 2028, 2032)

There are two occasions when extra care must be taken when using Table 3-1:

LEAP YEAR

In a leap year, if the time period includes the end of February, every date after February 28 increases its count by one.

Be careful when your calculations include a leap year or a period that covers the end of a year.

Write the day count on a diagram.

Example 3.2.4

How many days between January 15 and May 28, 2024? (2024 is a leap year.)

January 15 is the 15th day of the year from Table 3-1.

May 28 is the 148th day of the year from Table 3-1. However, since the time period includes February 29, you must add one day to the value shown in the table.

Therefore, the exact number of days = (148 + 1) − 15 = 134.

WHEN THE TIME PERIOD INCLUDES THE END OF THE YEAR

Example 3.2.5

How many days between October 15, 2021 and February 13, 2022? The technique is to subtract the first day value from 365 (to get to the end of the year) and then to add the second day value to that difference (which moves you into a second year).

October 15 is the 288th day of the year from Table 3-1. Therefore, the number of days to the end of the year is (365 – 288) or 77.

February 13 is the 44th day of the year from Table 3-1. Therefore, the exact number of days is 77 + 44 = 121. You can draw a time line with the exact dates shown. To assist in calculating the exact time, get into the habit of also writing the day count on the diagram.

This will automatically lead you to an exact count of the number of days between any two dates. To illustrate this, note:

178 – 44 = 134 days between the first two dates,

288 – 178 = 110 days between the second two dates, and

288 – 44 = 244 days between the first and last dates.

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?

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3.3 Calculating the Amount of Interest

We will apply the formula:

$$I = Prt$$

Example 3.3.1

1. How much interest must be paid on a $5,000 loan at 7.5% (per annum or pa) simple for 32 months?

Note: It is recommended that you **always** state the data before solving the problem.

$$P = \$5,000$$

$$r = 7.5\% \text{ or } 0.075$$

$$t = \frac{32}{12}\ years$$

Therefore:

$$I = \$5,000 \times 0.075 \times \frac{32}{12} = \$1,000$$

2. How much interest must be paid on a $6,500 loan at 8.5% (per annum or pa) simple if the loan was taken out on February 12, 2023 and repaid on August 15, 2023?

Using Table 3-1:

$$t = \frac{227 - 43}{365} = \frac{184}{365}\ year$$

Or using the BAII PLUS:

DT1 = 2.1223 [ENTER]

↓DT2 = 8.1523 [ENTER]

↓[CPT] DBD = 184

Therefore,

$$I = \$6,500 \times 0.085 \times \frac{184}{365} = \$278.52$$

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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3.4 Calculating Principal, Rate and Time

If any three of the four variables in the relationship ($I = Prt$) are given, you should be able to calculate the value of the unknown variable. Starting with the basic interest equation:

$$I = Prt$$

You can solve for P or *r* or *t* as follows:

$$P = \frac{I}{rt}$$

$$t = \frac{I}{Pr}$$

Since r is the rate in percent per year, the answer for t will be in years (or part thereof):

$$r = \frac{I}{Pt} \times 100$$

Since is the time in years, the answer for will be the annual rate of interest in decimal form.

Knowledge Check 3.1

Now try these exercises.

1. What is the interest charged to borrow $3,000 for 180 days at 6% simple interest?
2. If $55 interest was charged for a loan at 5.5% simple interest for 125 days, how much was borrowed?
3. How many days will it take a savings deposit of $900 to **earn** at least $65 interest, if the simple interest rate is 7.5%?
4. What rate of simple interest is used when a deposit of $975 earns $36.73 interest in 220 days?

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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3.5 Calculating the Future Value

The future value (or maturity value) is the total amount due at the end of a loan and is calculated by adding the interest due to the original principal.

$$FV = P + I$$

Example 3.5.1

1. If $9,600 is borrowed and $800 interest is due, what is the future value (maturity value) of the loan?

- P = $9,600
- I = $800
- FV = $9,600 + $800 = $10,400

2. Calculate the future value (maturity value) of a $6,500 loan at 8.5% (per annum or pa) simple if the loan was taken out on February 12, 2023 and repaid on August 15, 2023.

First we count the days:

Using the Calculator:

DT1 = 2.1223 [ENTER]
↓DT2 = 8.1523 [ENTER]
↓[CPT] DBD = 184

Using Table 3-1:

$$t = \frac{227 - 43}{365} = \frac{184}{365} years$$

Therefore,

$$I = Prt = \$6,500 \times 0.085 \times \frac{184}{365} = \$278.52$$

And

$$FV = P + I = \$6,500 + \$278.52 = \$6,778.52$$

Key Takeaways

Note: Some texts refer to the Future Value of a Simple Interest Problem as S, instead of FV.

You can simplify the Future Value equation as follows:

Since $I = Prt$ and $FV = P + I$

now, substitute for I. Therefore,

$$FV = P + (Prt)$$

Factoring out the P, you get

$$FV = P(1 + rt)$$

Your Own Notes

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3.6 Time Diagrams

The time diagram, or time line, graphically represents the time value of money.

The time diagram (Figure 3.6.1) is also called the time line . It is used to graphically represent the time value of money.

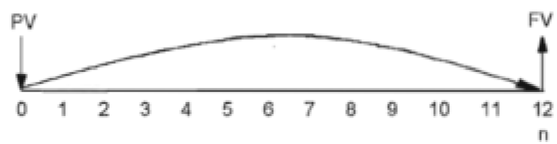


Figure 3.6.1: Time Diagram or Line

Although this diagram seems very simple and almost trivial at this point in the course, it is very important that you learn the basics here, and that you include the diagram in all problem solutions that involve cashflow.

Example 3.6.1

Calculate the maturity value of a 6-month loan of $5,000 at 8.25% simple interest.

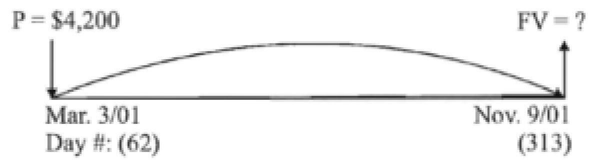
$$P = \$5,000; \ r = 8.25$$

Therefore:

$$FV = \$5,000(1 + 0.085 \times 0.5) = \$5,206.25$$

Example 3.6.2

Calculate the maturity value of a $4,200 deposit made at a simple interest rate of 6.75% pa from March 3, 2001 to November 9, 2001.



$$t = \frac{313 - 62}{365} = \frac{251}{365}\ years \text{ (from table 3.1)}$$

OR

Using the Calculator:

DT1 = 3.0301 [ENTER]

↓DT2 = 11.0901 [ENTER]

↓[CPT] DBD = 251

$$P = \$4,200;\ r = 6.75$$

Therefore:

$$FV = \$4,200\left(1 + 0.0675 \times \frac{251}{365}\right) = \$4394.95$$

Knowledge Check 3.2

Now try these examples (be sure to draw the time diagrams).

1. How much would have to be repaid if you borrowed $4,000 for 210 days at 8% simple interest?

2. Calculate the maturity value of $1,250 which was invested from March 10, 2022 to September 8, 2022 at 6.75% simple interest.
3. Joan Smith borrowed $2,500 from her father to start the school term. If she agreed to fully repay him the amount borrowed plus interest to be calculated at 3.75% simple, how much would she have to repay in exactly two years?

Your Own Notes

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https://pressbooks.bccampus.ca/businessmathematics/?p=247#h5p-1

3.7 Calculating Present Value

Key Takeaways

"Moving" money backwards in time.

Whenever interest is paid for the use of money, the value of the original principal will increase in relation to time. This concept is known as the time value of money.

Quite often, you need to calculate the Principal (when the future or maturity value, the rate and the time are known). You call this principal the present value. It can be found by rewriting the relationship as follows:

$$FV = P(1 + rt)$$

Therefore, dividing both sides by , you get:

$$P = \frac{FV}{1 + rt}$$

Example 3.7.1

Calculate the present value of a 6-month loan which requires a repayment of $6,500 including interest calculated at 8.25% pa simple interest.

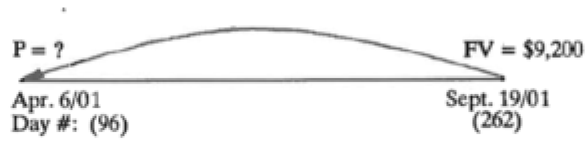
$$FV = \$6,500; \; r = 8.25$$

Therefore,

$$P = \frac{\$6,500}{(1 + 0.0825 \times 0.5)} = \$6,242.50$$

Example 3.7.2

How much should be invested on April 6, 2001 to amount to $9,200 (FV or maturity value) on September 19, 2001 at 8.5% simple interest?



$$t = \frac{262 - 96}{365} = \frac{166}{365}\ years \text{ (from table 3.1)}$$

OR

Using the Calculator:

DT1 = 4.0601 [ENTER]
↓DT2 = 9.1901 [ENTER]
↓[CPT] DBD = 166

$$FV = \$9,200;\ r = 8.5$$

Therefore,

$$P = \frac{\$9,200}{1 + 0.085 \times \frac{166}{365}} = \$8,857.59$$

Key Takeaways

When you use your calculator, be careful when you apply the ORDER OF OPERATIONS. Also, DO NOT ROUND OFF any intermediate values; instead use appropriate keystroke sequences or the calculator memory.

$$\frac{\$9,200}{1 + 0.085 \times \frac{166}{365}}$$

The following is a suggested sequence of keystrokes for the BAII Plus Calculator:

[166][÷][365][×][0.085][=][+][1][=][1/x][×][9200][=]

Key Takeaways

The two formulae, *FV = P(1+rt)* and *I = Prt*, interlink five variables, P, *r*, *t*, I and FV. Depending on the context any unknown of the five variables may be calculated if you know three of the variables. A useful way to deal with problems in this area is to write down the "known," and look for a formula which includes the "known" variables and the "unknown" variable.

Example 3.7.3

How long will it take to earn $50 interest if $1,000 is deposited at 6%?

$$I = \$50;\ r = 6$$

$$I = Prt$$

$$t = \frac{I}{Pr} = \frac{\$50}{\$1000 \times 0.06} = 0.833\ years$$

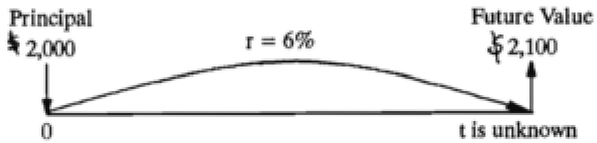
Note that t will be in "years" since the interest rate is understood to be "per year."

$$Actual\ time = 0.833\ years \times \frac{365\ days}{1\ year} = 304.2\ or\ 305\ days.$$

Now look at the same type of problem in a slightly different way:

Example 3.7.4

How long will it take $2,000 to accumulate to $2,100 if the simple interest rate is 6%?



$$FV = P(1+rt)$$

So:

$$1 + rt = \frac{FV}{P}$$

Then:

$$rt = \frac{FV}{P} - 1$$

And (given that $r \neq 0$):

$$t = \frac{\frac{FV}{P} - 1}{r} = \frac{\frac{\$2{,}100}{\$2{,}000} - 1}{0.06} = 0.833\ years = 305\ days$$

Knowledge Check 3.3

1. What rate of simple interest is used if a deposit of $2,000 amounts to $2,210 over 1.5 years?
2. What deposit will amount to $1,871.25 over a period of 33 months if interest is calculated at 9% simple?

Your Own Notes

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3.8 Time Value of Money

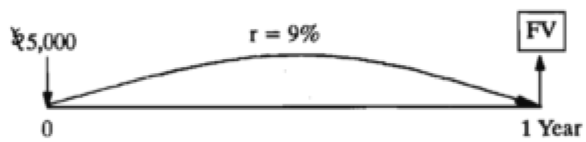
The value of money involves accounting/or interest over time.

INTRODUCTION USING FUTURE VALUE

The concept of **VALUE** of money involves accounting for interest over time.

Consider this question:

Would you rather have \$5,000 now or \$5,450 one year from now? Before you can answer this question, you must know what interest rate could be earned on the \$5,000 if you made that choice. Assume 9% simple interest, and calculate the future value.



$$FV = P(1 + rt)$$
$$= \$5,000(1 + 0.09 \times 1) = \$5,450$$

So, if you chose the \$5,000 now and deposited it at 9%, in one year you would have \$5,450, which "coincidentally" is the same as your other alternative. So there is no monetary difference between the two alternatives. The two **VALUES** are **EQUIVALENT.**

Key Takeaways

\$5,000 now and \$5,450 one year from now have **EQUIVALENT VALUES** AT 9%.

Remember that **VALUE** involves both interest rate and time. Try the following example Knowledge Check/

Knowledge Check 2.4

Would you rather have $10,000 now or $10,325 in 6 months time?

a. the interest rate is 7%.
b. the interest rate is 6%.

EQUIVALENT VALUES

Key Takeaways

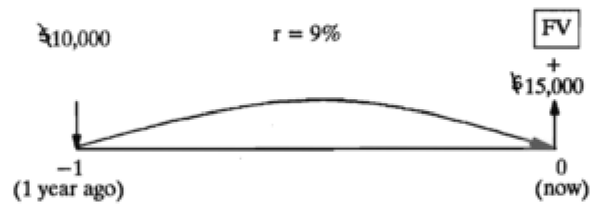
Do not add $ values at different points in time.

Now consider this problem:

Your company has borrowed $10,000 one year ago and $15,000 today. How much does the company owe? If you answered $25,000 you are making a common error. Dollars may **NOT** be compared (added or subtracted) when they are at different points in time. They may **ONLY** be compared by calculating **EQUIVALENT** values at the **SAME POINT** in time.

Example 3.8.1

Assume an interest rate of 9% and look at the problem again. (We are effectively saying that the $10,000 borrowed one year ago has a 9% simple interest obligation.) We can find the equivalent debt by calculating the equivalent value now of the $10,000 debt.

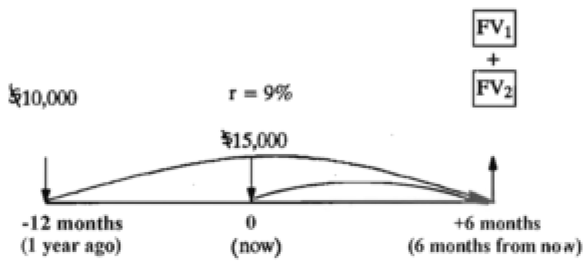


Now the two values may be added, since they are at the same point in time.

EQUIVALENT DEBT = $10,900 + $15,000 = $25,900

If both debts are settled **TODAY,** the company must pay $25,900.

How much must be paid to settle both debts **SIX MONTHS** from now?



If both debts are "moved" individually to the 1 year 6 month point they may be added.

$$FV_1 = \$10,000(1 + 0.09 \times 1.5) = \$11,350$$
$$FV_2 = \$15,000(1 + 0.09 \times 0.5) = \$15,675$$
$$Total\ Equivalent\ Debt = FV_1 + FV_2 = \$11,350 + \$15,675 = \$27,025$$

INTERPRETATION:

Each debt accumulates interest charges based on 9% per year, for the time the debt is outstanding.

Debt #1: $10,000 outstanding for 1½ years accumulates $1,350 interest charge and requires $11,350 repayment.

Debt #2: $15,000 outstanding for 6 months accumulates $675 interest charge and requires $15,675 repayment.

The total repayment is $27,025, which includes $2,025 total interest charges, at a point 6 months from now.

Knowledge Check 3.5

Debts of:

1. $20,000 borrowed nine months ago, and
2. $5,000 borrowed four months ago, and
3. $10,000 borrowed today

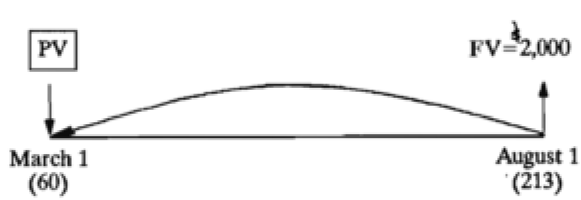
must be repaid, together with simple interest at 8%, 6 months from now. How much must be paid to settle all the debts?

PRESENT VALUE

In a similar manner to the “forward movement” along the time line, to obtain the future value, dollar amounts can be “moved backward” along the time line. The value of a **FUTURE** dollar amount at a point **EARLIER** in time is called the **PRESENT VALUE.** Usually this present value is calculated as the value **now** of an amount in the future, hence the name “Present” value. However, this is not a necessary condition. Any value, earlier in time than the dollar amount being considered is called “Present Value” at the point in time, and can be calculated in the same way.

Example 3.8.2

Find the value on March 1, 2021, of a repayment of $2,000 due on August 1, 2021.



Note: The day count number is in brackets.

To calculate the present value, a rate of interest, *r*, is required. The concept is that the present value can be considered as a deposit, at a given interest rate, which would amount to the $2,000 in the given time. For example, at an interest rate of 10%:

$$P = \frac{FV}{1+rt} = \frac{\$2000}{\left(1 + 0.10 \times \frac{153}{365}\right)} = \$1,919.54$$

Three interpretations of the answer are:

1. $1,919.54 is the present value of $2,000, 153 days ahead in time if the interest rate is 10%.
2. $1,919.54 now and $2,000 in 153 days have EQUIVALENT VALVES at 10%.
3. To accumulate $2,000 in 153 days at 10%, a deposit of $1,919.54 is required.

The wording of the three interpretations is different but they really all say the same thing monetarily.

Knowledge Check 3.6

Try these examples:

1. Find the value today of a promise to pay $5,000 1 year 7 months from now, if the simple interest rate is 9%.
2. Find the total equivalent debt today of two future debts.

   a. $2,000 in 7 months, and
   b. $4,000 in 13 months.

Use a simple interest rate of 7% to value the debts.

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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3.9 Equations of Value

Key Takeaways

All debts “moved” to date of final payment.

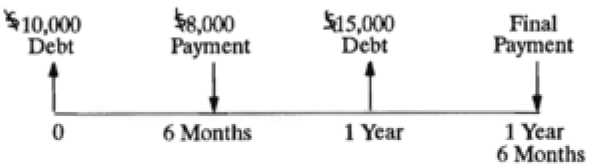
In many business situations, debts are not paid exactly as specified in contracts. Often a payment is made to partially offset a debt.
When two companies regularly carry out transactions with each other, there is often a running account arrangement, where debts and payments accrue over a period of time, with a final payment to settle the account.

Focal Dates

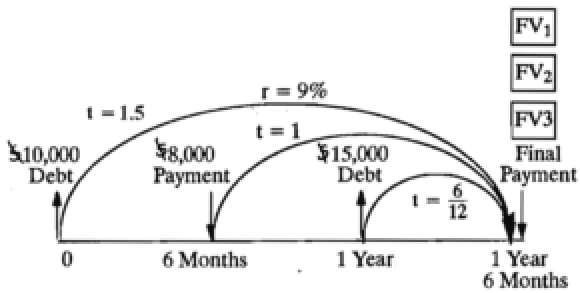
When there are multiple inflows and outflows, it is common to evaluate them all at a common date, often the date of the last payment. In this case, the time of the final payment is used as a **focal date**, and all debts and payments are “moved” to the focal date, and their EQUIVALENT VALUES calculated.

Example 3.9.1

Assume that we have two debts: $10,000 owed today and $15,000 owed one year from today. The company makes a payment of $8,000 six months from today and will make a final payment 1.5 years from today. The interest rate is 9%.



If the company wishes to settle the outstanding debt at the 1 year 6 month point, all the dollar amounts are “moved” to the 1 year 6 month point (the date of the final payment) where they can be added and subtracted. This point is called the **FOCAL DATE**.



First, calculate all FV. (EQUIVALENT VALUES)

$$FV_1 = \$10,000(1 + 0.09 \times 1.5) = \$11,350\ (value\ of\ first\ debt)$$
$$FV_2 = \$8,000(1 + 0.09 \times 1) = \$8,720\ (value\ of\ first\ payment)$$
$$FV_3 = \$15,000(1 + 0.09 \times 0.5) = \$15,675$$

Next, think about a balancing transaction at the focal date, where:

$$Value\ of\ all\ payments = Value\ of\ all\ debts$$
$$\$8,720 + LASTPAYMENT = \$11,350 + \$15,675$$
$$LASTPAYMENT = \$27,025 - \$8,720 = \$18,305$$

Key Takeaways

INTERPRETATION: On the **FOCAL** date, each debt will have accumulated interest **CHARGES**.

- The $10,000 debt is a debt of $11,350 on the focal date.
- The $15,000 debt is a debt of $15,675 on the focal date.
- The payment will have accumulated interest **CREDIT**, so that the $8,000 payment is worth $8,720 on the focal date.
- The total debt value on the focal date $11,350 + $15,675 = $27,025 is offset by the value of the payment, $8,720, so that the balance owing is $27,025 − $8,720 = $18,305. All payments and debts are “moved” to a focal date at the time of the final payment, and a payment−debt balancing EQUATION of VALUE is set up and solved for the unknown value.
  The effect of interest is calculated every time a payment is made.

Solving Equations of Value

Key Takeaways

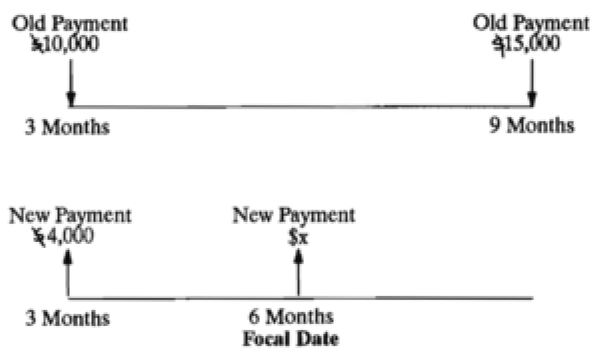
Balancing equation with a specified focal date.

The technique of "moving" all dollar amounts to a FOCAL DATE and setting up a balancing equation, can be used in a variety of situations where unknown values must be calculated. There are many occasions when the focal date may be established at ANY TIME by agreement between the creditor and the debtor.

Example 3.9.2

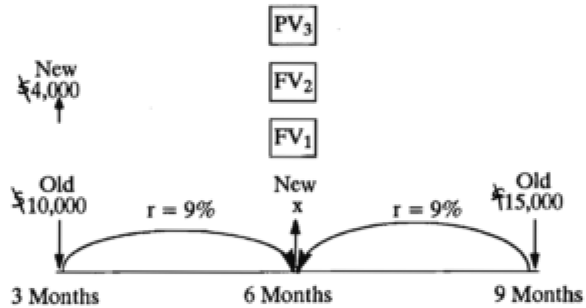
To settle a current debt your company has agreed to make payments of $10,000, 3 months from now and $15,000, 9 months from now. When the first payment becomes due, the company finds itself short of cash and enters into negotiation with the creditor. Eventually the creditor agrees that if the company makes a partial payment of $4,000 at the 3 month point, the balance must be paid at the 6 month point.

This is the situation:



Both debtor and creditor agree that the six month point will be the **FOCAL DATE** for calculations and that the payments will be valued using 9% simple interest. The two old payments are replaced by two new payments, with the second new payment, unknown, shown as *x*.
All dollar values are compared at the focal date.

Each dollar value is "moved" to the agreed focal date, and an **EQUATION OF VALUE** is set up to balance the values of old payments and new payments at the **FOCAL DATE**.



$$FV_1 = \$10,000\left(1 + 0.09 \times \frac{3}{12}\right) = \$10,225.00(old)$$

$$FV2 = \$4,000\left(1 + 0.09 \times \frac{3}{12}\right) = \$4,090.00(new)$$

$$PV_3 = \frac{\$15,000}{\left(1 + 0.09 \times \frac{3}{12}\right)} = \$14,669.93(old)$$

The FOCAL DATE EQUATION is:

$$\begin{aligned} \text{Value of new payments} &= \text{Value of old payments} x + \$4,090.00 = \$10,225.00 + \$14,669.93 \\ x &= \$24,894.93 - \$4,090.00 \\ x &= \$20,804.93 \end{aligned}$$

Now, let us look at a slightly more complicated set of payments. Assume that the agreement is for two payments, one at the 3 month point and one at the 6 month point as before, but with the second payment being exactly $15,000 more than the first payment. Now both new payments are unknown. However, they are related. If x is the first payment, then (x +$15,000) must be the second payment.

$$FV_1 = \$10,000[1 + (0.09 \times \frac{3}{12})] = \$10,225.00\ (old)$$

$$FV_2 = x[1 + (0.09 \times \frac{3}{12})] = 1.0225x\ (new)$$

$$PV_3 = \frac{\$15,000}{1 + (0.09 \times \frac{3}{12})} = \$14,669.93\ (old)$$

The EQUATION OF VALUE at the focal date is:

Value of new payments = Value of old payments

$$(x + \$15,000) + 1.0225x = \$10,225.00 + \$14,669.93$$

$$x + 1.0225x = \$10,225.00 + \$14,669.93 - \$15,000$$

$$2.0225x = \$9,494.93$$

$$x = \frac{\$9,494.93}{2.0225} = \$4,892.43$$

The first new payment (*x*) is \$4,892.43 and the second new payment is \$4,892.43 + \$15,000 or \$19,892.43.

Knowledge Check 3.8

Payments of \$3,500 at month 4 and \$5,500 at month 11 are to be replaced by a \$3,000 payment at month 6 and a final payment at month 12. The agreed simple interest rate is 8% and the agreed focal date is the date of the second payment. Find the size of the final payment.

Knowledge Check 3.9

Mr. M owes Ms. K a payment of \$10,000 that was due four months ago. He also owes her a second payment of \$9,000 due two months from today. The two parties negotiate a new payment arrangement whereby Mr. M will repay these debts with two equal installments -one due today and the other due six months from today . The agreed simple interest rate is 9% and the agreed focal date is today. Find the size of the two new equal payments.

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Solutions to Chapter 3 Knowledge Checks

Knowledge Check 3.1

1. P = \$3,000; r = 6% = 0.06; t =180/365 years

$$I = Prt = \$3,000 \times 0.06 \times \frac{180}{365} = \$88.77$$

2. I=\$55; r =5.5%=0.055; t =125/365 years $P = \frac{I}{rt} = \frac{\$55}{0.055\left(\frac{125}{365}\right)} = \$2920.00$

3. P = \$900, I = \$65, r = 7.5% = 0.075

$$t = \frac{I}{Pr} = \frac{\$65}{\$900 \times 0.075} = 0.962963 \text{ years}$$

To convert this answer to days, you must multiply by 365. To eliminate a rounding error, be sure to use the exact value from your calculator, i.e., just multiply the above value by 365 without re-entering the displayed value .

$$\therefore t = \frac{\$65}{\$900 \times 0.075} \text{years} \times \frac{365 \text{ days}}{1 \text{ years}} = 351.48 \text{ days}$$

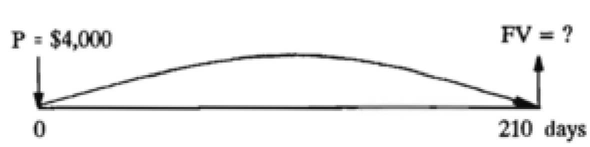
This should now be rounded up to 352 days.

4. P = \$975, I = \$36.73, t =220/365 years

$$r = \frac{I}{Pt} = \frac{\$36.73}{\$975\left(\frac{220}{365}\right)} = 0.062500932 = 6.25$$

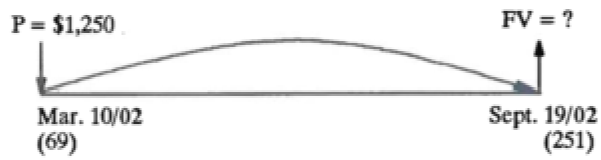
Knowledge Check 3.2

1. P = \$4,000, r = 8% = 0.08, t = 210/365 years



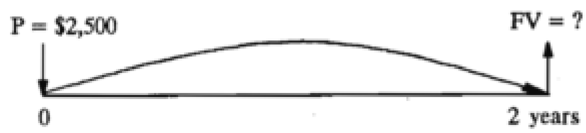
$$FV = \$4,000\left[1 + 0.08\left(\frac{210}{365}\right)\right]$$

2. P = \$1,250, r = 6.75% = 0.0675; $t = (251 - 69) + 365 = \dfrac{182}{365}$ years (from Table 3- 1)



$$FV = \$1,250\left[1 + 0.0675\left(\frac{182}{365}\right)\right] = \$1,292.07$$

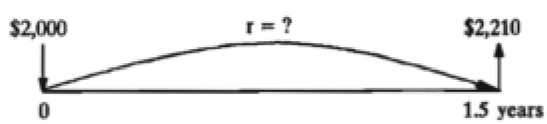
3. P = \$2,500, r = 3.75% = 0.0375, t = 2 years



$$FV = \$2,500(1 + 0.0375 \times 2) = \$2,687.50$$

Knowledge Check 3.3

1. P = \$2,000, FV = \$2,210, t = 1.5 years



Either of the two following approaches is acceptable:

Approach A:

$$I = FV - P = \$2,210 - \$2,000 = \$210$$

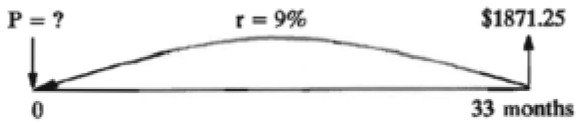
So

$$r = \frac{I}{Pt} = \frac{\$210}{\$2,000 \times 1.5} = 0.07 = 7$$

Approach B:

$$r = \frac{\frac{FV}{P} - 1}{t} = \frac{\frac{\$2{,}210}{\$2{,}000} - 1}{1.5} = 0.07$$

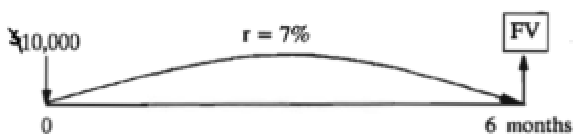
2. FV = $1,871.25, r = 9% = 0.09, t = 33 months =33/12 years.



$$P = \frac{FV}{1 + rt} = \frac{\$1,871.25}{1 + 0.09 \times \left(\frac{33}{12}\right)} = \$1,500.00$$

Knowledge Check 3.4

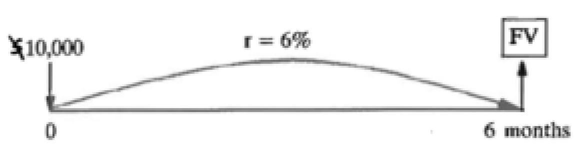
a. Find FV at 7%:



$$FV = P(1 + rt) = \$10,000\left(1 + 0.07 \times \frac{6}{12}\right) = \$10,350$$

Conclusion: The value of $10,000 now, in six months' time is $10,350. Since this is $25 greater than $10,325, you would prefer $10,000 now from a purely financial point of view.

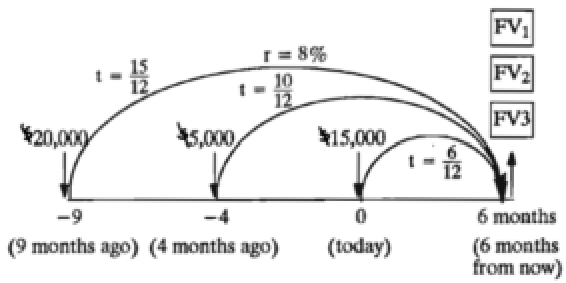
b. Find FV at 6%:



$$FV = P(1+rt) = \$10,000\left(1+0.06\times\frac{6}{12}\right) = \$10,300$$

Conclusion: The value of $10,000 now, in six months time is $10,300. Since this is less than $10,325, you would prefer $10,325 in six months time.

Knowledge Check 3.5

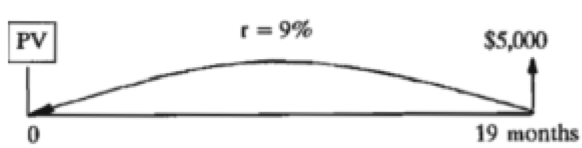


$$FV_1 = \$20,000\left(1+0.08\times\frac{15}{12}\right) = \$22,000.00$$
$$FV_2 = \$5,000\left(1+0.08\times\frac{10}{12}\right) = \$5,333.33$$
$$FV_3 = \$10,000\left(1+0.08\times\frac{6}{12}\right) = \$10,400.00$$

Total Debt Outstanding = $37,733.33(6 months from now)

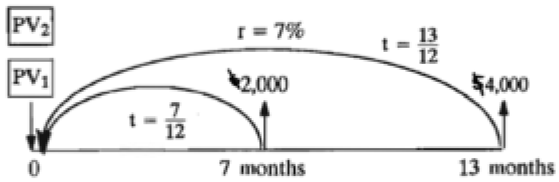
Knowledge Check 3.6

1.



$$PV = \frac{\$5,000}{1+0.09\times\frac{19}{12}} = \$4,376.37$$

2.

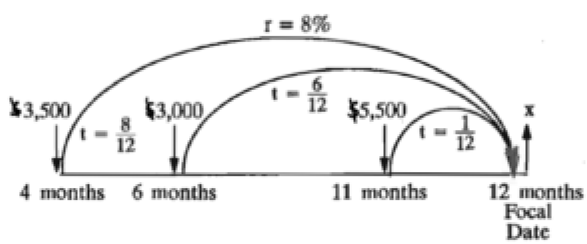


$$PV_1 = \frac{\$2,000}{1 + 0.07 \times \frac{7}{12}} = \$1,921.54$$

$$PV_2 = \frac{\$4,000}{1 + 0.07 \times \frac{13}{12}} = \$3,718.05$$

Total Equivalent Debt Now = $5,639.59

Knowledge Check 3.8



Value of "old" payments at the focal date :

$$\$3,500\left(1 + 0.08 \times \frac{8}{12}\right) + \$5,500\left(1 + 0.08 \times \frac{1}{12}\right)$$

$$= \$3,686.67 + \$5,536.67 = \$9,223.33$$

Value of "replacement" payments at the focal date:

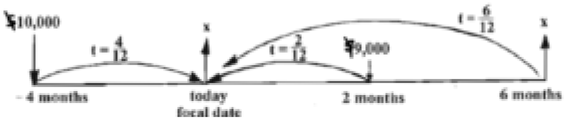
$$\$3,000\left(1 + 0.08 \times \frac{6}{12}\right) + x = \$3,120 + x$$

Therefore:

$$\$3,120 + x = \$9,223.33$$

$$x = \$9,223.33 - \$3,120.00 = \$6,103.33$$

Knowledge Check 3.9



Let the amount of two equal payments be x. Value of "old" payments at the focal date:

$$\$10,000\left(1+0.09\times\frac{4}{12}\right)+\frac{\$9,000}{\left(1+0.09\times\frac{2}{12}\right)}$$

Value of "replacement" payments at the focal date:

$$x+\frac{x}{\left(1+0.09\times\frac{6}{12}\right)}$$

And we can set these to be equal:

$$x+\frac{x}{\left(1+0.09\times\frac{6}{12}\right)}=\$10,000\left(1+0.09\times\frac{4}{12}\right)+\frac{\$9,000}{\left(1+0.09\times\frac{2}{12}\right)}$$

And solve, remembering to store all intermediate values in the calculator:

$$\begin{aligned} x+\frac{x}{1.045} &= \$10,000(1.03)+\frac{\$9,000}{1.015} \\ x\left(1+\frac{1}{1.045}\right) &= \$10,300+\$8866.995074 \\ x &= \frac{\$19,166.995074}{1.956937799}=\$9,794.38 \end{aligned}$$

Therefore: The size of the two equal payments is $9,794.38.

Chapter 3 Review Questions

Do not forget to draw your time diagrams!

1 For each principal, rate and time given below, compute the interest:

a. $2,500 at 14.2% for 1.5 years.
b. $3,200 at 8.75% for 16 months.
c. $8,300 at 11.2% for 160 days.
d. $800 at 13.6% for 212 days.

2 Calculate the interest for each of the following loans:

a. $850 at 11.5% from June 14, 2022 to October 19, 2022.
b. $2,800 at 11.25% from September 9, 20199 to March 19, 2020. (2020 was a leap year!)
c. $4,100 at 7.5% from July 15, 2022 to September 6, 2022.

3 Complete each row in the following table:

	Interest	**Principal**	**Rate**	**Time**
a.	?	$2,800	12%	210 days
b.	$461.25	$6,000	?	8 months
c.	$ 54.00	$1,440	11.5%	? days
d.	$ 81.30	?	6.25%	205 days

4 Find the interest rate which will pay $36.40 interest on a principal of $2,140 borrowed for 69 days.

[5] If a loan of $1,900 borrowed from October 22, 2021 to December 17, 2021 resulted in $33.85 interest, what was the simple interest rate charged?

[6] What principal will earn $95.20 if borrowed at 13.5% for 4 months?

[7] How many **days** will it take for a principal of $19,200 to earn $650.00 interest at 10%?

[8] What is the future value of $1,680 over 260 days at 11.25%?

[9] Find the principal and the interest if a loan at 12.5% for 9 months is completely paid off by the payment of $1,732.22 at the end of the 9 months.

[10] If 9 months interest at 8.725% is $186.20, what principal was borrowed?

[11] A loan at 9% was repaid by a payment of $3,710 of which $307.40 was interest. What was the length of time (in days) of the loan?

[12] If the future value of a loan for 222 days at 11.75% was $937.72, what was the principal of the loan?

[13] A loan is to be repaid in 9 months by a payment of $1,300. If interest is allowed at 13.15%, what is the present value of the loan?

[14] Payments of $5,000 due in 3 months and $6,000 due in 9 months are to be paid off with interest allowed at 13%. How much would be required to pay off the loan today? (Use today as the focal date.)

[15] LH should have paid a loan company $2,700 3 months ago and should also pay $\1,900

today. He agrees to pay $2,500 in 2 months and the rest in 6 months, and agrees to include interest at 11%. What would be the size of his final payment? Use 6 months as the focal date .

16 AW borrowed $9,000 on January 30, 2022 and agreed to pay 14% simple interest on the balance outstanding at any time. She paid $5,000 on March 9, 2022 and $2,500 on May 25, 2022. How much will she have to pay on June 30, 2022 in order to pay off the debt? Use June 30, 2022 as the focal date.

17 Debts of $8,000 due 8 months ago and $3,000 due in 4 months are to be paid off today with interest at 12%. Use today as a focal date and find the size of the payment.

18 $5,000 due today is to be paid instead by payments of $2,000 in 4 months and the balance in 9 months. Find the size of the last payment if interest is at 9% and the focal date is today.

19 Two payments of $1,200 each were due 30 and 60 days ago. They are to be paid off by two equal payments, one in 60 days and one in 90 days. If the focal date is 90 days from today and interest is at 12%, find the size of the payments.

20 Find the present values of the following payments if money is worth 8%:

a. $2,800 to be paid in 60 days.
b. $950 to be paid in 120 days.
c. $56,000 to be paid in 1 year.

21 You invest $1,000 for 4 years at 8% simple interest. How much interest will you earn?

22 You invest $6,000 for 2.5 years at 9% simple interest. How much interest will you earn?

23 $6,000 earns $180 in interest when invested for 30 months. What simple rate of interest is being paid?

[24] A $1,000 savings bond earns $600 in interest over the 12 years of the investment. What simple rate of interest is being paid?

[25] You would like to earn $1,000 in interest each year. If the interest rate is 6% simple how much money should you invest?

[26] You take a 3-year loan and repay the loan and $800 in interest. How much did you borrow if the interest rate was 10% simple?

[27] You would like to save for a vacation in Edmonton. You need $4,000 for your dream vacation. You deposit $3,000 in an account that pays 8% simple. How many months will it take you to save for your vacation if you make no other deposits?

[28] You invest $1,000 for 18 months at 8% simple interest. How much interest will you earn?

[29] You take out a loan for 400 days at 10% simple interest and at the end of that time you repay your loan plus $500 in interest. How much did you borrow?

[30] You invest $8,000 on March 3rd and withdraw the money on October 4th. If the interest rate is 9% simple, how much interest did you earn?

[31] You borrow $7,000 on August 16th and agree to pay back the loan plus interest calculated at 5% simple on June 15th of the next year (not a leap year). How much interest would you pay?

[32] You borrow $5,000 on June 15th and agree to pay back the loan plus interest calculated at 8% simple on March 31st of the next year (not a leap year). How much interest would you pay?

[33] You put $5,000 into a savings account earning 6% simple interest.

a. How many months will it take to for you to earn $75 of interest?
b. How many months will it take for your money to grow to $6,200?

34 You invest some money today at 4.5% simple interest for 120 days and the money grows to $7,408. How much did you invest today?

35 You invest $12,000 today into a fund that pays 6% simple. How much money will you have in 40 months time?

36 You borrow $6,000 to purchase a Jeep and agree to pay back all the money in 3.5 years. How much should you pay back if the interest rate is 12% simple?

37 You need $6,000 to return to school in 8 months time. How much should you invest today at 6% simple to achieve your goal?

38 A Freedom 35 financial planner claims you will need $1,175,000 to retire in 15 years time. How much should you invest today at 9% simple interest to reach your retirement goal?

39 How long will it take a sum of money to double if it earns 12% simple interest? (Answer in months)

40 You work as a real estate agent for Honest Dave's Realty Co. located in Burnaby. You have two debts corning due, one in six months for $5,000 and one in 12 months for $6,000. You recently sold a couple of houses and now have some extra cash. How much must you pay today to pay off both debts if interest is 6% simple? **Use today as your focal date.**

41 One of your customers has two debts outstanding, $600 is due 3 months from today and $900 was due 6 months ago. Instead, the customer would like to pay off both debts with a single payment one year from today. Calculate the size of that payment if interest is 12% simple. **Use one year from today as the focal date.**

42 You should have made two car payments of $1,000, 6 months ago and 3 months ago. The bank has agreed to let you repay the loan with equal payments in 3 and 6 months (from today). Calculate the size of these payments if interest is 14% simple. **Use 6 months as your focal date.**

43 You are attempting to repay your line of credit. One year ago you borrowed $5,000 and 6 months ago you borrowed $4,000. You have examined your cash flow projections and decide to repay the line of credit with two payments in 12 and 18 months. The second payment will be $2,000 larger than the first. Find the size of the payments **using 18 months as your focal date.** Interest is 6% simple.

44You have borrowed from your line of credit. 6 months ago you borrowed $5,000 and today you borrowed $15,000 . You plan to pay off the entire line of credit with three equal payments at 3, 5 and 8 months (from today). Find the size of each payment if your bank charges you 9.75% simple interest? **Use today as the focal date.**

45 Repeat the previous problem using five months as the focal date. (By comparing the answers to both questions you will see that it depends slightly on the focal date chosen -but only for simple interest.)

46 You were supposed to make a payment of $3,500 three months ago and a second payment of $6,100 five months from today. Instead you have arranged with the bank to make a payment one month from now and a second payment, half as large, 6 months from today. Calculate these payments if the bank charges 8.25% simple interest. **Use the date of the first unknown payment as the focal date.**

47 You borrowed $1,000 on November 30th and another $1,500 on December 31st. Your arrange with the bank to pay the entire amount on February 15th of the following year. If the interest is 12% simple (per annum) how much must you pay on February 15th? Use February 15th as the focal date.

48 You are considering purchasing a car. The owner has offered to let you make two payments of $4,000 each with the first payment at 6 months and the second payment at 10 months.

Instead, you would like to make a payment of $4,000 in 8 months and pay the rest today. Find the size of today's payment if the interest rate is 6% simple. **Use today as your focal date.**

[49] You have two debts coming due. A $1,500 debt is due in 15 months and another debt for $1,000 is due in 33 months. Instead, you would like to repay the debts with two equal payments at 3 and 9 months. Find the size of the equal payments if interest is calculated at 6% simple per year. **Use 9 months as your focal date.**

Notes

1. a. $532.50
   b. $373.33
   c. $407.50
   d. $63.19
2. a. $34.01
   b. $165.70
   c. $44.65
3. a. interest = $193.32 b. rate= 11.53125%c. time= 119.02 or 120 days d. principal = $2,316.06
4. 8.9977% or 9%
5. 11.6121%
6. $2,115,56
7. 123.5677 or 124 days
8. $1,814.63
9. Principal= $1,583.74 Interest= $148.48
10. $2,845.46
11. 366.3898 or 367 days
12. $875.17
13. $1,183.30
14. $10,309.59
15. $2,335.58
16. $1,770.03
17. $11,524.62
18. $3264.68
19. $1,247.11

20.
   a. $2,763.66
   b. $925.65
   c. $51,851.85

21. $320
22. $1,350
23. 1.2%
24. 5%
25. $16,666.67
26. $2,666.67
27. 50 months
28. $120
29. $4,562.50
30. $424.11
31. $290.55
32. $316.71
33. a. 3 months b. 48 months
34. $7300 exactly
35. $14,400
36. $8,520
37. $5,769.23
38. $500,000
39. 100 months
40. $10,514.75
41. $1,716
42. $1,103.19 each
43. First payment: $4,054.19; second payment: $6,054.19
44. 7,038.53
45. $7,036.45
46. $6,426.52 and $3,213.26 in 6 months
47. $2,548.00
48. $3,846.87
49. $1,157.23

Chapter 4: Compound Interest

In the previous chapter, you saw that, in *simple interest* calculations, the amount of interest earned is the *same* for every time period – for example, the amount earned during the first 30-day period is the same as the amount earned in the second 30-day period.

In long-term loans, an increase in the amount of interest paid should follow an increase in the loan's value.

In long-term loans, it is felt that, since the value of the loan increases with time (because of interest), the amount of *interest should increase* in later time periods. This increase is accomplished by calculating the interest as compound interest.

4.1 Introduction to Compound Interest

Compound interest is the interest paid on previously earned interest as well as on the original principal.

Example 4.1.1

Consider a loan of $1,000 for two years at 10% per year interest. If simple interest is used, the lender would receive at the end of the loan.

$$\$1,000 \times (1 + 0.1 \times 2) = \$1,200$$

But at the end of the first year, the lender could reasonably consider the value of the loan to be

$$\$1,000 \times (1 + 0.1 \times 1) = \$1,100$$

and feel that the interest for the second year should be based on this amount as principal. The lender would then receive

$$\$1,100 \times (1 + 0.1 \times 1) = \$1,210$$

at the end of the second year. This is described as interest *compounded annually,* or *converted* (changed to principal) *annually.*

Compounding takes effect in accounts where interest is regularly added to the balance.

While it is possible to consider the principal of a loan as continuously increasing because of interest, the usual way to state and calculate compound interest is to follow the idea above and increase the principal by the interest at regular intervals. The interval to be used is stated in the rate. Commonly used intervals are given by the terms:

- Annually
- semi-annually (every half year = every six months)
- quarterly (every three months)
- monthly
- biweekly (every two weeks or fortnight)

Occasionally loans are compounded weekly or daily.

The interest rate in the example on the previous page would be described as 10% compounded annually. Compounding takes effect in accounts in which the interest is added to the balance at regular intervals.

Example 4.1.2

Imagine a trust account set up with an initial balance of $5,000 and an earning of 8% compounded semi-annually. After two years the account would look like this:

Time	Interest	Balance
0		$5,000.00
6 months	$200.00	$5,200.00
1 year	$208.00	$5,408.00
18 months	$216.32	$5,624.32
2 years	$224.97	$5,849.29

The interest calculations for the account would be as follows: At 6 months:

$$Interest = \$5,000 \times 0.08 \times 0.5 = \$5,000 \times 0.04 = \$200$$

Note: 8% compounded semi-annually means 4% every half year. At 1 year:

$$Interest = \$5,200 \times 0.04 = \$208$$

At 18 months:

$$Interest = \$5,408 \times 0.04 = \$216.32$$

At 2 years:

$$Interest = \$5,624.32 \times .04 = \$224.97$$

The total interest is $849.29 vs $800 at 8% simple interest.

Knowledge Check 4.1

Try to perform these fundamental operations of compound interest. Complete the following account table, using an interest rate of 16% compounded quarterly.

Time	Interest	Balance
0		$8,000.00
3 months	$320.00	$8,320.00
6 months	$332.80	?
9 months	?	$8,998.91

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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4.2 Nominal and Periodic Rates

Key Takeaways

Nominal rate*: Percent annual rate;* ***Periodic rate****: Rate that gives percent interest each period.*

When compound-interest rates are given as above, the percent annual rate is called the *nominal rate* and denoted by *j*. To show clearly what the compounding interval is, add a subscript to the *j* (j_m) . This indicates the *number of times per year* the interest is to be compounded.

- Thus, 8% compounded semi-annually would be j_2 = 8%, m=2.
- Thus, 18% compounded monthly would be j_{12} = 18%, m=12.

The periodic rate is used in most BAII Plus calculations.

The rate that gives the percent interest each period is called the *periodic rate* and is denoted by *i*. This is the rate *actually used* in most formulas. The following are examples.

Example 4.2.1

1. For 8% compounded semi-annually (j_2 = 8% =0.08), $i = \dfrac{0.08}{2} = 0.04 = 4$
2. For 18% compounded monthly (j_{12} = 18% =0.18), $i = \dfrac{0.18}{12} = 0.015 = 1.5$

In general, to get *i* from j_m:

$$i = \frac{j_m}{m}$$

Your Own Notes

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4.3 Compound Interest Formula

The procedure for adding interest each period can always be used to find the future value of a loan or deposit, but the following general formula gives the future value more directly.

$$FV = PV(1+i)^n = PV\left(1+\frac{j_m}{m}\right)^n$$

where:

- FV = future value of the loan
- PV = present value of the loan (principal)
- *i* = periodic interest rate
- *n* = number of compounding periods

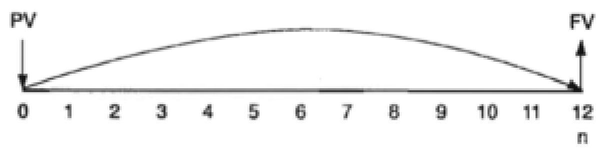

Figure 4.1: Cash Flow

Example 4.3.1

To see how the formula is developed, consider the $5,000 loan at 8% compounded semi-annually for two years.

$$\text{First, } i = \frac{0.08}{2} = 0.04$$

The balances would be:

At 6 months:

$$\$5,000(1+0.04) = \$5,000(1.04) = \$5,200$$

At 1 year:

$$\$5,200(1.04) = \$5,000(1.04)(1.04) = \$5,000(1.04)^2 = \$5,408$$

At 18 months:

$$\$5,408(1.04) = \$5,000(1.04)^2(1.04) = \$5,000(1.04)^3 = \$5,624.32$$

At 2 years:

$$\$5,624.32(1.04) = \$5,000(1.04)^3(1.04) = \$5,000(1.04)^4 = \$5,849.29$$

This last calculation for the two-year balance is the general formula for FV with:

- PV = \$5,000
- i = 0.04
- n = 4 = 2× 2

In general, the values for i and n are found by:

$$i = \frac{j_m}{m}$$

and

$$n = (\text{number of years}) \times (\text{number of periods per year}) = t \times m$$

Knowledge Check 4.2

Use the compound-interest formula to find the following future values:

1. The future value of a deposit of \$8,000 at 16% compounded qua1terly for nine months.
2. The future value of a loan of \$1,000 for two years at 10% compounded annually.
3. The future value of a loan of \$2,500 for four years at 8% compounded monthly.

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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4.4 Present Value

For many investment decisions, it is necessary to find the principal, or present value, that corresponds to a given future value.

Example 4.4.1

Consider a note that will pay $10,000 to whoever owns it three years from now. If an investor wants to earn 10% compounded annually, what is the most he or she should pay for the note?

You have:

- i=10%=0.10 per year
- n=3 years
- FV= $10,000

Thus, using the compound-interest formula:

$$\begin{aligned} PV(1+i)^n &= FV \\ PV(1.10)^3 &= \$10,000 \\ PV &= \frac{\$10,000}{1.1^3} = \$7,513.15 \end{aligned}$$

This result can be checked by accumulating the money in an account, as shown in the next box.

Time	**Interest**	**Balance**
0		$7,513.15
1	$751.32	8,264.46
2	826.45	9,090.91
3	909.09	10,000.00

FORMULA FOR PRESENT VALUE

The compound-interest formula can be rewritten to give the present value directly. Divide both sides of the equation by $(1+i)^n$

$$\frac{PV(1+i)^n}{(1+i)^n} = \frac{FV}{(1+i)^n}$$

Cancel and rewrite:

$$PV = \frac{FV}{(1+i)^n}$$

Knowledge Check 4.3

Find the present values by using the formula for PV given above.

1. The present value of $5,849.29 due in two years if interest is at 8% compounded semi-annually.
2. The present value of $8,998.91 due in nine months if interest is at 16% compounded quarterly.
3. The principal of a loan that would amount to $50,000 in six years at 8.5% compounded annually.

Your Own Notes

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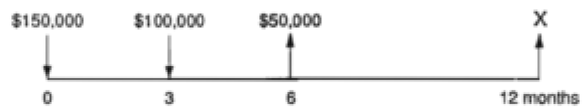
4.5 Multiple Cash Flows

Up to this point, you have considered situations that have only a single loan or deposit and a repayment. There are, however, many cases in which there are many inflows and/or outflows to a particular individual or company. For example, consider the company that is developing a business and needs cash inflows (such as loans) early in its life and will balance these inflows with outflows later.

Example 4.5.1

Suppose, for instance, that a builder plans to finance a project through a bank and will borrow $150,000 now and $100,000 in three months, then repay $50,000 **in** six months and the rest in one year's time. Interest is to be paid on the outstanding balance at 12% compounded monthly.



Then you can find the size of the last payment (denoted by x above) by following through an account and adding interest every month.

Cash inflows to the builder are positive.

Time (months)	Interest	Cash Flow	Balance
0	0.00	$150,000.00	$150,000.00
1	1,500.00	0.00	151,500.00
2	1,515.00	0.00	153,015.00
3	1,530.15	100,000.00	254,545.15
4	2,545.45	0.00	257,090.60
5	2,570.91	0.00	259,661.51
6	2,596.62	-50,000.00	212,258.12
7	2,122.58	0.00	214,380.70
8	2,143.81	0.00	216,524.51
9	2,165.25	0.00	218,689.76
10	2,186.90	0.00	220,876.65
11	2,208.77	0.00	223,085.42
12	2,230.85	0.00	225,316.27

A shorter way to check the account is to accumulate the future value on the *outstanding balance* only *at the time of cash flows.* For the above account, then, the result is as follows:

Time (months)	Interest	Cash Flow	Balance
0		$150,000	$150,000
3	$\$150,000(1.01)^3 = \$154,545.15$	100,000.00	254,545.15
6	$\$254,545.15(1.01)^3 = 262,258.12$	-50,000	212,258.12
12	$\$212,258.12 \times (1.01)^6 = 225,316.27$	-225,316.27	0

Your Own Notes

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4.6 Equivalent Values

We say that the payments of $50,000 and $225,316.27 at the given times are *equivalent* to the loans of $150,000 and $100,000 at their given times.

It is often the case that there are two sets of flows. (In the builder's case mentioned previously, the $150,000 and $100,000 inflows form one set; the $50,000 and $225,316.27 outflows form the other.)

Two sets of cash flows are *equivalent in value* whenever they have the *same value after interest* is allowed for.

If you proceed through an account with two sets of equivalent cash flows – one viewed as inflows to the account, the other as outflows – the final balance must be $0.

Key Takeaways

In an equation of value, the values of inflows, after interest, are equal to the values of outflows.

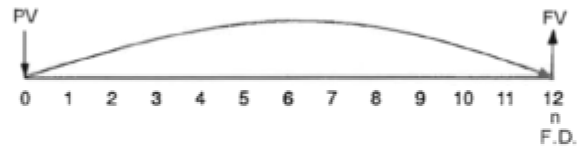
Evaluating payments by following through an account is fundamental to understanding equivalence, but this method is difficult to apply to more complex cash flows where the cash flow to be determined occurs earlier, or where there are several cash flows to be determined. In such cases, the following *equation of value method* is preferred.

An *equation of value* make the values of *inflows equal* to the values of *outflows* after interest is allowed for – that is, after allowing for the time value of money. This is accomplished by finding the value of every cash flow at some fixed date, the ***Focal Date.*** The results obtained will be the same as those found by calculating the values by an account, as shown above.

The compound-interest formula,

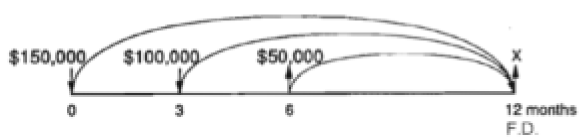
$$FV = PV(1+i)^n$$

is in itself a simple equation of value, with the focal date at the nth period



Example 4.6.1

The builder's situation discussed above could be analyzed by an equation of value and corresponding cash-flow diagram, as follows:



For a focal date of one year, the equation is:

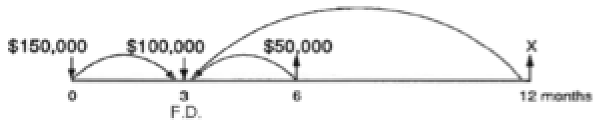
$$\begin{aligned}
\text{Value of inflows} &= \text{Value of outflows}\\
(\text{FV of } \$150,000) + (\text{FV of } \$100,000) &= (\text{FV of } \$50,000) + x\\
\$150,000(1.01)^{12} + \$100,000(1.01)^{9} &= \$50,000(1.01)^{6} + x\\
\$169,023.75 + \$109,368.53 &= \$53,076.01 + x\\
\$278,392.28 - \$53,076.01 &= x\\
x &= \$225,316.27
\end{aligned}$$

Through the example above, you will appreciate that the equation of value allows for the same interest as the account method does. This is true no matter what focal date is chosen – which is not the case in Simple Interest.

Since the time of the last flow was chosen as the focal date, all flows are evaluated by future value. Any cash flow that *precedes* the focal date is evaluated by finding its future value. Any cash flow that *follows* the focal date is evaluated by finding its present value.

To see how that works, study the next example, which uses a focal date of three months for the problem above.

Example 4.6.2



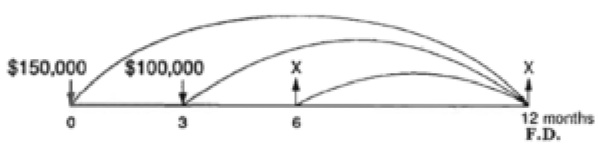
$$
\begin{aligned}
\text{Value of inflows} &= \text{Value of outflows} \\
(\text{FV of } \$150,000) + \$100,000 &= (\text{PV of } \$50,000) + (\text{PV of } x) \\
\$150,000.00(1.01)^3 + \$100,000 &= \frac{\$50,000}{1.01^3} + \frac{x}{1.01^9} \\
\$154,545.15 + \$100,000 &= \$48,529.51 + \frac{x}{1.01^9} \\
\$254,545.15 - \$448,529.52 &= \frac{x}{1.01^9} \\
x &= 1.01^9(\$206,015.64) \\
x &= \$225,316.27
\end{aligned}
$$

This is the same answer as that obtained before, but stating and solving the equation are slightly more complex. Choosing the best focal date is a matter of convenience.

Problems may have several unknown cash flows, but it must be possible to state them in terms of one variable, since there is only one equation.

Example 4.6.3

As an example, suppose, in the builder's case, the loans were to be repaid with two equal-sized payments: one at six months, the other at one year.



The equation is simplest if you choose one year as the focal date.

Then:

$$\begin{aligned}
\text{Value of inflows} &= \text{Value of outflows}\\
(\text{FV of }\$150,000) + (\text{FV of }\$100,000) &= (\text{FV of }x) + x\\
\$150,000(1.01)^{12} + \$100,000(1.01)^{9} &= x(1.01)^{6} + x\\
\$169,023.75 + \$109,368.53 &= 1.061520151x + x\\
\frac{\$278,392.28}{2.061520151} &= x\\
x &= \$135,042.23
\end{aligned}$$

To check that this is correct, enter the values in an account as shown in the box:

Time (months)	Interest	Cash Flow	Balance
0		$150,000	$150,000
3	$\$150,000(1.01)^{3} = \$154,545.15$	100,000.00	254,545.15
6	$\$254,545.15(1.01)^{3} = \$262,258.12$	-135,042.23	127,215.89
12	$\$127,215.89(1.01)^{6} = \$135,042.23$	-135,042.23	0

It is *easy to check* results by an account, but it would be difficult to solve such a problem by an account. And in this case an equation would have to be solved at some point.

Your Own Notes

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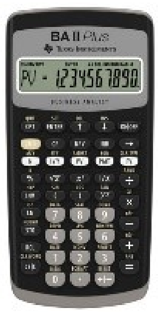
4.7 Compound Interest with the BAII Plus

Using Financial Calculator Functions



The financial calculator recommended for this course is the BAII Plus. Both this and other financial calculators have built-in compound-interest functions. It is possible to do almost all of the course calculations to the same accuracy without these functions, but the process is much faster if they are available.

The functions you will use in this chapter are controlled by the following keys:

P/Y and C/Y	**N**	**I/Y**	**PV**	**PMT**	**FV**
How many times do we compound per year?(m)	*Number of periods*	*Nominal Interest Rate, j_m*	*Present Value*	*0 (for now)*	*Future Value (One of PV and FV is negative!)*

Key Takeaways

Financial Calculators should have built-in compound-interest functions.

In the same row is the PMT key which you will use in the next chapter. For this chapter, the PMT value should be set at 0. It's always best practice to set it to 0 each and every time!

Example 4.7.1

Invest \$100 at j_2 =6% for 4 years. N = 2× 4 = 8 periods.

Step	**To**	**Press**	**Display**
1	Clear previous saved values (except *P/Y* and *C/Y*)	[2ND] [CLR TVM]	
2	Enter *N=8* periods	[N] [8]	N = 8
3	Enter nominal interest rate, *I/Y* = 6%. *(Annual interest rate in percentage)*	[I/Y][6]	I/Y = 6
4	Select *P/Y* and *C/Y* worksheet	[2ND] [P/Y]	
5	Set number of payments per year, *P/Y* = 2	[ENTER] [2]	P/Y = 2
6	Set Number of compounding periods per year, *C/Y*=2 *(By default, C/Y is set as the same as P/Y)*	[↓] [2] [ENTER]	C/Y = 2
7	Return to standard calculator mode	[2ND] [QUIT]	0

Step	To	Press	Display
8	Enter present value, *PV* =100	[1][0][0][±][PV]	PV = 100
9	Enter periodic payment, *PMT* =0	[0][PMT]	PMT = 0
10	Compute future value, *FV* *(positive value for inflow)*	[CPT][FV]	FV = -126.6770081

We write this as:

P/Y	C/Y	N	I/Y	PV	PMT	FV
4		4×2=8	6	+100	0	CPT: -125.6770

Leaving some spaces for Annuities, in Chapter 5.

Example 4.7.2

To illustrate the use of the financial calculator, suppose you want to obtain the future value of a $5,000 loan at 8% compounded semi-annually for two years.

P/Y	C/Y	N	I/Y	PV	PMT	FV
2		2×2=8	8	5,000	0	**CPT**: -5,849.29

You will see the answer, $5,849.29, which was obtained earlier in the chapter by an account and by the formula. Note that the answer appears as a *negative* value on the calculator. This is because the calculator performs an *equation of value* in the form of:

$$\text{Value of Inflows} + \text{Value of Outflows} = 0$$

Hence it must make either inflows or outflows negative. (Since PV was made positive, it must make FV negative.)

From now on, you will normally indicate the procedure for solving problems – especially if they are likely to be done with computer functions – by listing the available values of the variables and what is required.

The answer would be negative on the calculator, but this will be mentioned only if confusion may arise from the answer.

With the calculator functions, any one of the functions N, I/Y, PV, or FV can be found from the others. How this is done is illustrated in the next example, which uses some previous problems.

The calculator assumes each problem has a cash outflow (entered as a negative) and a cash inflow (entered as a positive). For simplicity, we will always show PV as positive, and FV as negative.

Example 4.7.3

You borrow $1,000 and agree to repay the loan with a single payment in 2 years. How much should you pay if interest is charged at 8% compounded quarterly?

P/Y	C/Y	N	I/Y	PV	PMT	FV
4		4×2=8	8	1,000	0	**CPT**: -1,171.66

To look at values entered in your calculator, just press [RCL] and then the value you want to check, e.g., [RCL] [N] should show 8.

Example 4.7.4

If an invested \$8,000 results in a future value of \$8,998.91 in nine months, what is the interest rate compounded quarterly?

You have:

P/Y	C/Y	N	I/Y	PV	PMT	FV
4		4× 9/12 =3	**CPT**	8,000	0	-8,998.91

Answer: 16% compounded quarterly.

Alternatively, you could solve the algebra problem:

$$\$8,000(1+\frac{j_m}{4})^3=\$8,998.91$$

Which simplifies to:

$$j_m=4\left(\sqrt[3]{(\frac{FV}{PV})-1)}\right)=4\left((\frac{FV}{PV})^{1/3}-1\right)$$

But this is a much tougher problem!

Example 4.7.5

If \$150,000 is invested at 12% compounded monthly and results in a future value of \$169,023.75, for how long must it have been invested?

P/Y	C/Y	N	I/Y	PV	PMT	FV
12		**CPT**	12	150,000	0	-169,023.75

Answer: 11.9999973 or 12 months.

Alternatively, we could solve the algebra problem:

$$\$150,000\left(1+\frac{0.12}{12}\right)^n = \$169,023.75$$

Which simplifies, using logarithms to:

$$n = \log_{1.01}\left(\frac{\$169,023.75}{\$150,000}\right)$$

In general, the calculator is a very good option – you do not need to use logarithms, and can solve much faster.

Knowledge Check 4.5

1. Find the future value of a loan of $12,000 for 16 months at 15% compounded monthly. In doing this, you should write down the values entered into the TVM:

P/Y	C/Y	N	I/Y	PV	PMT	FV

3. How much must be invested at 11% quarterly to get $9,500 in two years?

P/Y	C/Y	N	I/Y	PV	PMT	FV

4. If a bank deposit of $80,000 amounts to $84,934.22 after gaining interest compounded monthly for one year, what was the nominal rate per month?

P/Y	C/Y	N	I/Y	PV	PMT	FV

Your Own Notes

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4.8 Equivalent and Effective Rates

Interest rates are ***EQUIVALENT*** if they provide the *same amount of interest* on any loan. Consider the two rates:

1. 20.5% compounded semi-annually.
2. 20% compounded quarterly.

If you take *any principal* and *any length of time*, you will find that the two rates always result in exactly the same future value – hence the *same interest*. This is because they are related by the algebraic expression:

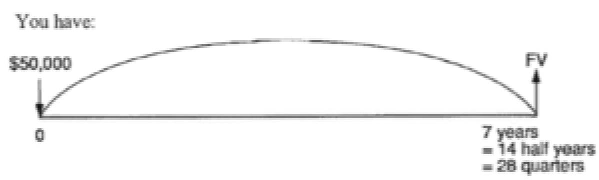
To check the equivalence, consider the following example.

$$\left(1+\frac{0.205}{2}\right)^2=\left(1+\frac{0.20}{4}\right)^4$$

Example 4.8.1

Suppose $50,000 is invested for seven years at the interest rates noted above. Find the future value of the $50,000 for each interest rate.

You have:



Key Takeaway

If two rates produce the same result for any principal and time, the rates will do so for any values.

For *rate* a: 20.5% compounded semi-annually

- n = 7×2 = 14 half-years
- I = $\frac{0.0205}{2} = 10.25$
- PV =$50,0000
- FV = ?

Answer: $196,006.46 for *rate b*

That exactly the *same future value* is obtained for both rates bears out the claim that the rates are equivalent. In fact, if two rates produce

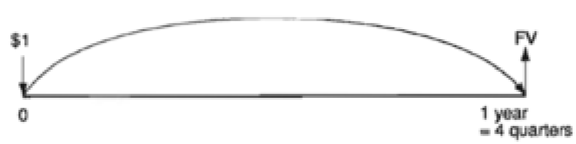
the same result for *any* (non-zero) principal and time, then the rates will do so for *any values.* Hence they are equivalent.

You can use calculator functions to find equivalent rates fairly easily, but first we will use the Future Value formula. You can use any size of investment and any length of time, but to illustrate this in the next example, $1 for one year is used.

Example 4.8.2

Suppose you are given the 20% compounded quarterly rate mentioned above and are asked to find the equivalent rate compounded

semi-annually. You are given:



- n = 1× 4 = 4
- $i = \frac{0.20}{4} = 0.05$
- PV = $1
- FV = ?

Using the Future Value Formula, we have;

$$FV = PV(1+i)^n = \$1(1.05)^4 = \$1.21550625$$

(Leave this answer in the calculator.)

For the new rate, the *only thing* that will be different (aside from i) is that it is to be compounded only twice in the year.

So we have

- n = 1× 2 =2
- i = ?
- PV = $1
- FV = $1.21550625

Using the Future Value Formula, we have;

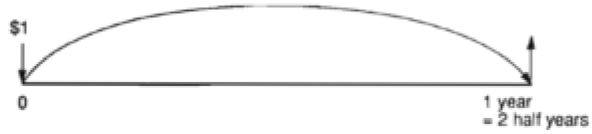
$$\begin{aligned} PV(1+i)^n &= FV \\ \$1(1+i)^2 &= \$1.21550625 \\ (1+i)^2 &= \frac{\$1.21550625}{\$1} \\ 1+i &= \sqrt{\$1.21550625} = 1.1025 \\ i &= 1.1025 - 1 \\ &= 0.1025 \end{aligned}$$

So the nominal rate would be j_2 = 10.25% × 2 = 20.5%.

EFFECTIVE RATES

The equivalent rate compounded annually for a given compound interest rate.

Compound-interest rates are *compared* by finding for each rate the equivalent rate *compounded annually*. For a given compound-interest rate, the equivalent rate compounded annually is called its ***effective rate***.



Example 4.8.3

Find Effective rates for the following:

a. 20.5% compounded semi-annually: $j_2 = 0.205$ so $i = 0.1025$

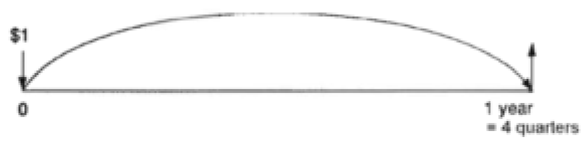
$$\$1(1.1025)^2 = \$1.21550625$$

To find the effective rate, we can just evaluate the Future Value equation for n = 1 year:

$$\$1(1+i)^1 = \$1.21550625$$

This is trivial to solve: $j_1 = i =$ 21.550625%. In fact, the interest earned on $1 invested for a year is the equivalent rate!

b. 20% compounded quarterly.



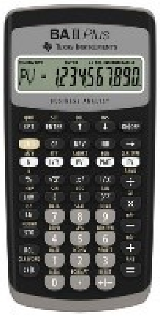
$n = 1 \times 4$ quarters, $i = 20 \div 4 = 5\%$, PV = $1, FV =?

$$\$1(1.05)^4 = \$1.21550625$$

So we can see that the effective rate is also j_1 = 21.550625%. You can see, then, that each rate was equivalent to 21.550625% compounded annually – which also shows that they were equivalent, since they were both *equivalent* to the *same effective rate*.

EFFECTIVE RATES WITH THE BAII PLUS

Example 4.8.4



A bank offers a certificate that pays a nominal interest rate of 15% with quarterly compounding. What is the annual effective interest rate?

Step	To	Press	Display
1	Select Interest Conversion worksheet	[2ND] [ICONV]	NOM = 0
2	Enter nominal interest rate, *NOM* = 15	[1][5][ENTER]	NOM = 15
3	Enter number of compounding periods per year, *C/Y* = 4	[↓] [4] [ENTER]	C/Y = 4
4	Compute annual effective rate, *EFF*	[↓][CPT]	EFF = 15.8650415

Example 4.8.5

Try the following: ↓

Find the interest rate, compounded quarterly, that is equivalent to 15% compounded monthly.

[2ND][ICONV]

NOM = 15 [ENTER]

↑C/Y = 12 [ENTER]

↑EFF = [CPT] 16.07545177

↓C/Y = 4 [ENTER]

↓NOM = [CPT] 15.18828125

Thus, the following three rates are all equivalent:

$$j_{12} = 15\% \Leftrightarrow j_1 = 16.07545177\% \Leftrightarrow j_4 = 15.18828125\%$$

Your Own Notes

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4.9 Fractional Periods

Occasionally it may be necessary to deal with compound interest for a fraction of a period, for example taking money out of a savings account after two weeks. In these cases, it is important to understand what the policy of the bank or lender is.

For a savings account that pays interest monthly you may still receive interest for the two weeks (at the end of the month) if they pay interest on the daily balance but pay it monthly. In other words, some banks calculate interest based on the equivalent daily rate (j_{365}) , but pay monthly. The interest rate paid depends on the total daily closing balance. Interest rate is applied to the entire balance, calculated daily, and paid monthly.

However, for GICs if you cash them in early you may get no interest or if it is a redeemable GIC you will get interest for the time that you had it. So if you have a 3-year cashable GIC that pays interest annually and you cash it in after 10 months it is unclear whether they will use simple interest or fractional exponents. We reached out to a bank to ask – the head office of TD Bank has stated that they had no idea, they said the “computer calculates it”.

For such cases, unless otherwise stated, use the compound-interest formula:

$$FV = PV(1+i)^n$$

with n having a fractional part. The following example justifies this procedure.

Example

Suppose that $1,000 was borrowed at an interest rate of 10% compounded annually, and originally was scheduled to be repaid after two years. Instead, it was decided to repay the loan after 1.5 years.

1. **Both parties have agreed to have interest calculated for the partial period**. At that time, by the compound-interest formula, the amount to be repaid would be: $FV = \$1000(1.10)^{1.5} = \$1,000(1.15369\ldots) = \$1,153.69$. Now suppose this amount were to be reinvested at the same rate for the remaining half year. In this case, the money accumulated would be:$FV = \$1,153.69(1.10)^{0.5} = \$1,210.00$, which is exactly what would have resulted from the original two year loan: $FV = \$1,000(1.10)^2 = \$1,210.00$
2. **Both parties have not agreed to have interest calculated for the partial period.** In this case, we would round down, so $n = 1$, and we receive no interest for the

final year. $FV = \$1,000(1.10)^1 = \$1,100.00$. In this case, we get a lot less interest!

In fact, part 1 gives us a general property of compound interest: If the balance is found at any time and *reinvested* at the same rate, nothing changes. You may find that some financial institutions prefer to deal with the fractional period by assuming that simple interest is paid for that portion. At one time this method was popular because of the difficulty of performing the calculations for the formula without calculators or computers. It also meant that the institutions would receive more money since, for a partial period, simple interest is slightly higher than compound interest.

Key Takeaways

General property of compound interest: If the balance is found at any time and reinvested at the same rate, nothing changes.

Knowledge Check 4.4

A loan for $6,000 will be taken out for four years at 14% compounded semi-annually. However, it is decided that the money should be repaid after three years and two months.

a. Find the accumulated amount to be repaid.
b. Check to see that reinvesting this amount for the remaining 10 months would produce the same amount as the original four-year loan.

Your Own Notes

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4.10 Average Rates

Often, interest rates change over the life of an investment. In this case, we often try to find the average rate. To illustrate, here is an example:

Example 4.10.1

7 years ago, Warren's grandmother gave each of her grandchildren different amounts of money. Warren invested $4,000 of his money with CJC investments. For the first year, CJC put his money in a GIC which paid an effective rate of 5.9%. For the next 2 years, CJC moved the money into Canada Savings Bonds, which paid out a semi-annual rate of 7.8%. To capitalize on the markets, CJC moved the money into a mutual fund, where it earned an average monthly interest rate of 4.8% for three years. For the final year, the money was put it into stocks which paid out 5.6% compounded quarterly. How much money does Warren have today?

To solve this, we will take the Present value, and find its value at each point:

YEAR	P/Y	C/Y	N	I/Y	PV	PMT	FV
1	1		1	5.9	4,000	0	CPT: -4236
2-3	2		2 ×2 = 4	7.8	4236	0	CPT: -4936.488637...
4-6	12		12× 3=36	4.8	4936.488637...	0	CPT: -5699.43497...
7	4		4	5.6	-5699.43497...	0	CPT: 6025.36864...

We can also look at this algebraically:

$$FV = \$4,000(1.059)^1(1.039)^4(1.004)^{36}(1.014)^4 = \$6025.36864...$$

So Warren has $6,025.37.

Now we can work out what Warren's average effective rate is. We take the PV and FV, and use the calculator to find out what effective rate will turn $4,000 to $6025.37 in 7 years. Make sure to use your memory buttons to store all of the decimal points.

P/Y	C/Y	N	I/Y	PV	PMT	FV
1		7	CPT	4,000	0	-6025.36864..

We get an effective rate of $j_1 = 6.027290102\%$. This is Warren's average rate.

Once we have the effective average rate, we can use it to solve other problems:

Example 4.10.2

Amy also invested with CJC investments, who invested her money in the same manner. If Amy now has $8150, how much did she invest 7 years ago with CJC?
To solve this, we can use the average rate already computed:

P/Y	C/Y	N	I/Y	PV	PMT	FV
1		7	6.027290102	CPT	0	-8,150

Which tells us that Amy invested $5,410.46, 7 years ago.

Your Own Notes

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Chap 4 Knowledge Check Answer Key

Knowledge Check 4.1

Time	Interest	Balance
0		\$8,000.00
3 months	\$320.00	\$8,320.00
6 months	\$332.80	\$8,652.80
9 months	\$346.11	\$8,998.91

Knowledge Check 4.2

1. \$8,000(1.04)^3 = \$8,998.91
2. \$1,000(1.10)^2 = \$1,1210
3. \$2,500(1.0066666667)^{48} = \$3,439.17

Knowledge Check 4.3

1. $\dfrac{\$5,849.29}{1.04^4} = \$5,000$

2. $\dfrac{\$8,998.91}{1.04^3} = \$8,000$

3. $\dfrac{\$50,000}{1.085^6} = \$\$30,647.25$

Knowledge Check 4.4

a. With i = 0.07, n = 3 × 2, FV = $9,209.76

b. n=(l0+12)× 2, FV=$10,309.11,
when PV = $6,000, n = 4 × 2, FV = $10,309.12

The difference is due to rounding off the FV in (a).

Knowledge Check 4.5

1. $14,638.67

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12		16	15	12,000	0	**CPT**

2. $7,646,61

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	4		16	15	CPT	0	-9,500

3. 6% nominal

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12		12	**CPT**	80,000	0	-84,934.22

Review Questions for Chapter 4

1 Complete the following table, assuming an interest rate of 10% compounded quarterly.

Time	Interest	Balance	
0	0	$8,000.00	(PV)
3 months 6 months	$200.00	8,405.00	
9 months			
12 months		8,830.50	(FV)

2 Make a table with the same headings as previous, but for a loan of $12,000 for 5 months at 15% compounded monthly.

3 Write the following nominal rates in the "j_m" notation and find the corresponding periodic rates, i (include the period with these). The first line is completed as an example.

Nominal Rate	**Frequency**	**Nominal Rate, j_m**	**Periodic rate, *i***
8%	semi-annually	j_2= 0.08	i =0.04
15%	monthly	?	?
10%	quarterly	?	?
9%	annually	?	?
10.4%	weekly	?	?

4 Write each of the following rates as nominal rates and complete the table.

Nominal Rate	**Compounding Frequency**	**Nominal Rate, j_m**	**Periodic Rate, *i***
?	?	j_1 = 7%	?
?	?	j_4 = 9%	?
?	?	j_{12} = 6%	?
?	Quarterly	?	
?	Monthly	?	
?	Semi-annually	?	

5 Complete the following table, using the compound-interest formula to calculate the future value of each loan.

PV	Interest Rate	Length	*i*	*n*	FV
$11,500	9% compounded quarterly	2 years	?	?	?
$7,400	6% compounded annually	4 years	?	?	?
$14,000	8.6% compounded monthly	8 months	?	?	?

6 Government compound-interest savings bonds have the interest compounded every year. Suppose that a $1,000 bond paid interest at 8.5% compounded annually and was kept for three years. Find the value (Future Value) of the bond at the end of the three years by:

a. Showing the interest and balance each year.
b. Using the compound-interest formula to get FV at the end of two years.

7 A loan of $24,000 for two years is to carry interest at 14% compounded semi-annually. Use the compound-interest formula to find the value of the loan at the end of two years.

8 A "junk" bond was supposed to pay interest at 25% paid annually. Unfortunately, no payments were made for the last seven years, so the interest was allowed to compound at 25% compounded annually.

a. Find the value at the end of each of the seven years for a bond with a principal of $1,000 at the start.

b. Also find the value the loan would have had at the end of each year if the interest had been simple interest at 25% annually. Note the difference caused by compounding.

9 Find the present values of each of the amounts below, filling in the rest of the table when you do so.

PV	Interest Rate	Length	*n*	FV
?	6.5% compounded semi-annually	2 years	?	$11,364.76
?	16% compounded monthly	3.5 years	?	$25,000.00
?	I0% compounded quarterly	1 year	?	$30,000.00

Check your last result by assuming the PV was invested for one year in an account paying 10% compounded quarterly and adding on the interest each period as in Problem 1.

10 According to the terms of his uncle's will, Tom Jones is to receive $50,000 2.5 years from now. Tom would like to borrow as much as he can now and pay it off with his inheritance. Use the compound-interest formula to find out how much he can borrow if the interest rate is as follows:

a. 10% compounded monthly.
b. 9% compounded monthly.
c. 8% compounded monthly.

11 Ann Lee has a lease on a government property which must be paid by a lump sum of $6,000 every year. The next payment is due in 10 months from now, and Ann plans to invest enough money in an account at her bank so that the amount in the account in 10 months will cover the $6,000 payment. If the interest rate is 8.5% compounded monthly, how much should she place in the account?

12 Ajax Company is borrowing $100,000 now and has agreed to pay 11% compounded annually on the outstanding balance at all times. Ajax plans to pay $30,000 at the end of the first year of the loan and $35,000 at the end of the second year. The remaining debt is to be completely paid off by a single payment at the end of the third year. Draw a cash-flow diagram and find the amount that should be paid at the end of the third year by:

a. Finding the interest and balance due at the end of each year.
b. Using an equation of value at a focal date (e.g., at year 3).

[13] AC Holdings has taken over a company with a debt that was to be paid by a payment of $85,000 one year from now. Instead, AC has agreed with the holder of the debt to pay it off early and be allowed 12% compounded quarterly for early payment. It plans to pay $40,000 now and the rest in six months.

a. Draw a cash-flow diagram and find the amount that should be paid in 6 months.
b. Repeat (a) but assume the payments now and in 6 months are to be equal in size.

[14] North Credit Union advertises that it will pay 7% compounded annually on money on deposit for periods longer than one year and that the money may be taken out, with interest, at any time after the first year. A depositor placed $20,000 on deposit at the above rate for four years, but decided to withdraw it after 2.5 years.

a. How much should the depositor receive?
b. Show that, if this amount were deposited at the same rate for the remaining 1.5 years, the result would be the same as if it had never been withdrawn.

[15] A loan of $17,000 for three years resulted in a future value of $21,560.11. Use your calculator functions to find the nominal interest rate if the loan was compounded:

a. quarterly
b. semi-annually.

[16] Use your calculator functions to find the FV of a principal of $1.00 invested for one year at 16% compounded quarterly, and leave the results in your calculator. Then find the nominal monthly rate that will give the same FV – the rate compounded monthly that is equivalent to 16% compounded quarterly. Check your answer by finding the future value of a principal of $10,000 invested for two years at each rate. You should get the same answer in each case.

[17] Complete the following table of equivalent nominal rates, giving answers in percent to six decimals. Each row will contain equivalent rates.

Effective Rate, j_1	j_2	j_4	j_{12}
?	?	?	j_{12} = 16%
?	?	j_4 = 12%	?
?	j_2 = 9%	?	?

[18] Use your results in Row 1 in the previous problem to find the future value of a loan of $4,000 for 18 months by doing the compounding:

a. monthly.
b. Quarterly.
c. Semi-annually
d. Annually

[19] Your first child was born this year and you decide to save for her education. You deposit $2,000 into an RESP (registered education savings plan) that pays 4.5% compounded annually (j_1=0.045). How much will your child have in 18 years? Use both the formula and the TVM Keys.

[20] You borrow $5,000 from a private loan company and agree to pay it back with interest calculated at 10% compounded quarterly. How much will you owe in 30 months? Use both the formula and the TVM Keys.

[21] You are expecting a tax refund of $3,000 six months from now. You take your T4 slips to H&P Square Tax preparation service and they agree to give you the money now if you sign over the refund to them. How much money will you receive today if interest is calculated at j_{12} = 9$%. Use both the formula and the TVM Keys.

[22] You go to purchase a brand new motorcycle and the dealer quotes you a price of $14,000 to be paid as a single payment in 30 months. You would like to pay cash today for the motorcycle. How much should you offer if interest is calculated at 8% compounded semi annually? Use the formula and the TVM Keys.

[23] You have $75,000 and decide to purchase a 30 year Government of Canada strip bond with an interest rate of 7.0% compounded semi-annually. How much will the bond be worth in 30 years? (i.e., what is the maturity value of the strip bond?)

[24] Financial planners predict you will need one million dollars to retire.

a. You would like to buy a 30-year strip bond with an interest rate of 7% compounded semi-annually that has a maturity value of one million dollars. How much will you need to pay for this strip bond today?
b. Unfortunately, you only have $75,000 available to buy a bond. You are considering buying a junk bond (risky bond) because you know that the interest rates are much. higher than for a Government of Canada bond. What nominal interest rate, compounded semi-annually, do you require so that the bond will have a maturity value of $1,000,000 in 30 years?
c. Your friend convinces you that junk bonds are too risky to be used as a retirement investment. If instead, you buy the Government of Canada bond (with your $75,000) that pays 7% compounded semi-annually, how many years will it take you to reach your goal of having $1,000,000?

[25] If an investment grows from $10,000 to $16,000 in 27 months, what was the nominal rate of interest, compounded quarterly?

[26] If an investment grows from $4,000 to $6,000 in 48 months, what was the nominal rate of interest:

a. compounded monthly?
b. compounded quarterly?
c. compounded semi-annually?
d. compounded annually?

[27] You would like to save to return to school. You deposit $4,000 into a GIC that pays . You have decided to return to school when your savings grows to at least $6,000. If you make no more contributions, how many years will it take you to reach your goal?

[28] How many years will it take $300.00 to accumulate to $425.29 at 7% compounded monthly?

29 An investment of $1,500.00 made 27 months ago is now worth $1753.48. What nominal rate of interest, compounded quarterly, did this investment earn?

30 You have always wanted to go to Orlando, Florida. You estimate you will need $6,000 for your trip. You deposit your tax refund of $5,016.10 into an account that pays 12% compounded monthly. How many years will it take to reach $6,000?

31 You put $5,000 into a 3-year term deposit that pays interest at 6.3% compounded quarterly. After the end of the three years you renew the term deposit plus accumulated interest at 7.2% compounded semi annually for an additional three years.

a. How much money will you have at the end of the six years?
b. How much interest did you earn?

32 You borrowed some money from the Honest Shark Finance Co. You made only one payment of $25,292.49 at the end of 5 years to pay off the loan. The interest rate was 18.5% compounded semi annually for the first 2 years and 19.6% compounded quarterly for the remaining years.

a. How much did you borrow?
b. How much interest did you pay?

33 You are going to purchase a car and are given two options to pay.

- You can pay $18,000 in cash today, **or**
- $5,000 after one year, and a second payment of $15,000 in 2 years (from today).

Which option is better? Your answer should be stated in terms of today's dollars. The prevailing interest rate is 7% compounded monthly (j_{12} = 7%)?

34 You have a line of credit that charges interest at j_{12} = 8%. You borrowed $4,000 six months ago and $2,000 two months ago. You would like to repay the loan with a single payment in 6 months time. Calculate the size of the payment. **Use 6 months as the focal date.**

35 Repeat the above question with you making two equal payments at six and twelve months. Find the size of the equal payments. **Use 6 months as the focal date. Note:** To save time use some of the values you calculated above since the focal date is the same.

36 You purchased a machine for your plant and the contract calls for equal payments of $12,000 in 12 months and 24 months. Your cash flow is better than you projected so you would like to repay the loan early with a single payment today. Calculate the size of the payment if interest is 9% compounded semi annually. **Use today as the focal date.**

37 Repeat the above question with you making two equal payments, one today and one in six months time. Find the size of the equal payments. **Use today as the focal date. Note:** To save time use some of the values you calculated above since the focal date is the same.

38 You were supposed to pay $5,000 today. Up until yesterday you had the money but you lost it all gambling at the Great American Casino. You arrange with the bank to defer the payment. You will pay $2,000 at the end of 18 months and 3 equal payments at the end of 24 months, 30 months and 36 months . The interest rate is 10% compounded quarterly (j_4 = 10%).

a. Find the size of the equal payments. Use 30 months as the focal date.
b. How much interest did you pay as a result of gambling and losing the $5,000?

39 You borrowed $12,000 2.5 years ago. You agreed to repay the loan with one payment 15 months from today, and a second payment, $3,000 larger than the first, 27 months from today. Find the size of each payment if money is worth 11% compounded quarterly? **Use 27 months as the focal date.**

40 You have $10,000 to invest today. How much would you have (to the nearest $100) at the end of 30 years if your money earns:

a. 12% interest, compounded monthly?
b. 12% simple interest?

41 Your Credit Card charges you a nominal interest rate of 18.6% per year. If they compound interest monthly, what effective rate of interest do they charge?

42 You are offered two options for your mortgage.

- A Canadian bank offers you a rate of j_2 = 8.40%
- A US bank offers you j_{12} = 8.30%.

Which rate is better? Convert both rates to effective rates to compare them.

43 What is the effective rate of interest earned on an investment of $10,652,952,497,853.65 that earns 15% compounded quarterly?

44 What nominal rate, compounded semi-annually, is equivalent to 8% compounded quarterly?

45 What nominal rate, compounded quarterly, is equivalent to 8% compounded monthly?

46 What is the effective interest rate of 8% compounded monthly?

47 Canada Premium Bonds are a new kind of Canada Savings Bond. They were available for sale until November 1, 2022. They proposed to pay the following interest rates, compounded annually.

Year 1	2.50%
Year2	3.00%
Year3	4.00%
Year4	4.85%
Year 5	6.00%

What effective interest rate would a Canada Premium Bond purchaser average over the five years? **Hint:** use the formula to find the FV. Do not round the FV.

48 What nominal rate, compounded monthly, is equivalent to 10% compounded semi-annually?

49 What is the effective interest rate of 20% compounded quarterly?

50 You have $5,000 saved. You are considering two investments:

- Canada Savings Bonds (CSB) which pay 5.25% in the first year, 6% in the second year, and 6.75% in the third year, compounded semi-annually.
- A 3-year "Bond-Beater" Guaranteed Investment Certificate (GIC) offered by a bank that pays 5.75%, 6.5%, and 7.25% compounded annually in the three successive years.

a. How much would you have at the end of 3 years if· you bought a $5,000 CSB and how much if you bought a $5,000 GIC?
b. Find the average effective interest rate earned for the entire 3-year period for both the CSB and the GIC.

51 You invest $6,000 in a mutual fund. Your investment of $6,000 earns the following returns.

Year	**Return**
Year 1	$j_2 = 10\%$
Year 2	$j_{12} = 6\%$
Year 3	$j_4 = 8\%$
Year 4	$j_1 = 7\%$

a. How much would your $6,000 investment be worth at the end of the fourth year?
b. How much did you earn (in dollars) during those four years?
c. In the fifth year the mutual fund loses money and the value of your investment decreases by $400. Calculate the average annual rate of return, compounded semi-annually, for the five years you have held your investment.
d. A friend has invested in a different mutual fund and says she doubled her money in seven years. What effective interest rate did she average over the seven years?

[52] A Canada Savings Bond pays j_2 = 5% in the first year, j_2 = 8% in the second year, and j_2 = 10% in the third year. What nominal interest rate, compounded quarterly, would provide the same return? Do not round the FV.

[53] An investment earned 12%, compounded quarterly, for two years and 10% compounded annually for the next three years. Calculate the average annual rate of return, compounded monthly, for the five years. Do **not** round the FV.

[54] An investment earned 20%, 15%, -10%, 25%, and -5% in 5 successive years. What average annual rate of return, compounded annually, was earned for the entire 5-year period? **Hint:** use the formula to find the FV. Do not round the FV.

Notes

1.

Time	**Interest**	**Balances:**
0m		$8,000.00
3m	$200.00	$8,200.00
6m	$205.00	$8,405.00
9m	$210.13	$8,615.13
12m	$215.37	$8,830.50

2.

Time	Interest	Balances:
0m		$12,000.00
1m	$150.00	$12,150.00
2m	$151.88	$12,301.88
3m	$153.77	$12,455.65
4m	$155.70	$12,611.35
5m	$157.64	$12,768.99

3.

Nominal Rate	Frequency	Nominal Rate, jm	Periodic rate, i
8%	semi-annually	j_2 = 0.08	i =0.04
15%	monthly	j_{12} = 0.15	i = 0.0125
10%	quarterly	j_4 = 0.10	i= 0.025
9%	annually	j_1 = 0.09	i= 0.09
10.4%	weekly	j_{52} = 0.104	i= 0.002

4.

Nominal Rate	**Compounding Frequency**	**Nominal Rate, j_m**	**Periodic Rate, *i***
7%	annually	j_1 = 7%	i= 0.07
9%	quarterly	j_4 = 9%	i = 0.0225
6%	monthly	j_{12} = 6%	i = 0.005
12%	quarterly	j_4 = 12%	i = 0.03
24%	monthly	j_{12} = 24%	i = 0.02
17%	semi-annually	j_2 = 17%	i = 0.085

5.

PV	Interest Rate	Length	*i*	*n*	FV
$11,500	9% quarterly	2 years	0.0225	8	$13,740.56
$7,400	6% annually	4 years	0.06	4	$9,342.33
$14,000	8.6% monthly	8 months	0.007166	8	$14,823.09

6.

Year	Interest	Balances
0		$1,000.00
1	$85.00	$1,085.00
2	$92.23	$1,177.23
3	$100.06	$1,277.29

7. $31,459.10

8.

a.

Year	Balance
0	$1,000.00
1	$1,250.00
2	$1,562.50
3	$1,953.13
4	$2,441.41
5	$3,051.76
6	$3,814.70
7	$4,768.38

b.

Year	Balance
0	$1,000.00
1	$1,250.00
2	$1,500.00
3	$1,750.00
4	$2,000.00
5	$2,250.00
6	$2,500.00
7	$2,750.00

9.

PV	Interest Rate	Length	n	FV
$11,500	6.5% compounded semi-annually	2 years	8	$11,364.76
$7,400	16% compounded monthly	3.5 years	4	$25,000.00
$14,000	I0% compounded quarterly	1 year	8	$30,000.00

10.
a. $38,980.40
b. $39,959.34
c. $40,963.72

11. $5,591.10

12.

Year	Balance
1	$81,000.00
2	$54,910.00
3	$60,950.10

b. $100,000 (1.11)^3 = $30,000 (1.11)^2 + $35,000 (1.11) + *x* x = $60,950.10

13. a. $37,684.65 b. $38,876.54

14.
1. $23,685.88
2. FV = $ 23,685(1.07)^1.5 =$ 20,000(1.07)^4 =$ 26,215.92

15.
a. 8% compounded quarterly
b. 8.08% compounded semi-annually

16. 16% compounded quarterly = 15.7913% compounded monthly

17.

Effective Rate, j_1	j_2	j_4	j_{12}
17.227080%	16.214281%	16.542910%	16%
12.550881%	12.180000%	12%	11.881961%
9.202500%	9%	8.900966%	8.835748%

18.
a. $5,076.94
b. $5,076.94
c. $5,076.94
d. $5,076.94 (using a fractional period)

19. $4,416.96
20. $6,400.42
21. $2,868.47
22. $11,506.98
23. $590,856.82

24.
a. $126,934.31
b. 8.82331%
c. 37.65 years

25. j_4 = 21.44411%

26.
a. j_{12} = 10.17956%
b. j_4 = 10.26616%
c. j_2 = 10.39790%

d. j_1 = 10.66819%

27. 5.5 years
28. 5.0 years
29. 7.0000%
30. 1.5 years
31. \$7,457.11 (the interest earned is \$2,457.11)
32. \$10,000 (the interest paid is \$15,292.49)
33. In today's dollars the 2nd option is worth \$17,708.60 which is less than \$18,000 so the second option is cheaper
34. \$6,441.19
35. \$3,284.78
36. \$21,051.50
37. \$10,757.36
38. a. Payment= \$1,396.46
    b. the interest paid is \$1,189.38
39. First payment: \$8,083.05; second payment \$11,083.05
40. a. \$359,500 b. \$46,000
41. j_1 = 20.2705% (**effective** means compounded annually)
42. Canada: j_1= 8.57640%; US: j_1 = 8.62314%, Canadian bank is the better deal
43. j_1 = 15.86504%
44. j_2 = 8.08%
45. j_4 = 8.05345%
46. j_1 = 8.29995%
47. j_1 = 4.062377967%, **not** 4.07%
48. j_{12} = 9.79782%
49. 21.550625%
50. a. CSB: \$5,970.10 GIC: \$6,039.45
    b. CSB: j_1= 6.08904%; GIC: j_1 = 6.49825%
51. a. \$8,134.05
    b. \$2,134.05
    c. j_2 = 5.14246%
    d. j_1 = 10.40895137%
52. j_4 = 7.58458%
53. j_{12} = 10.49364%
54. 8.08142% compounded annually

Chapter 5: Annuities

An *annuity* is a series of payments, with one payment per period for a given number of periods.

The *variables* in an annuity are:

- N: the number of periods, in the *term* (equal to the number of payments)
- I/Y: the nominal interest rate
- PMT: the periodic payment
- PV: the present value of the annuity
- FV: the future value of the annuity

Annuities can be used for:

- *amortizing* a loan, where the loan value is the *present value* of the annuity. The variables are N, I/Y, PMT and PV.
- *accumulating* an amount, where the amount is the *future value* of the annuity. The variables are N, I/Y, PMT and FV.

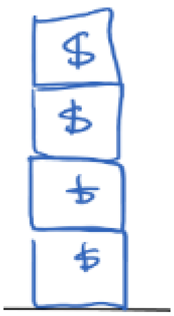
Annuities can be solved in many ways:

- by formula
- by calculating interest period by period
- by focal date methods
- by calculator program, which is what we will concentrate on.

5.1 Adding to a Savings Account

Learning Outcomes

Calculate the amount saved and interest earned when regular deposits are made.



You make a series of equal-sized payments (deposits), at regular intervals over the course of a fixed time period. When you make these regular deposits, you are adding to the money in the account and building up a positive account balance. Therefore, both PV (your initial deposit) and PMT (your subsequent regular deposits) must have the same sign (positive). The future value (FV) must have the opposite sign (negative)[1]:

PV	Interest	PMT	FV
Initial Deposit	**+ % Gain**	**+ Regular Deposits**	**= Final Withdrawal**
0 or +	**+**	**+**	**–**

To make sense of FV (your final withdrawal) being negative, you could imagine that we are closing the account at the end of the investment period. When we close it, we *withdraw* all of the funds, making the final/future value opposite in sign from the regular deposits (PMT) and the initial amount deposited (PV).

See the sections below for key formulas, tips and examples related to calculations when adding to a savings account.

CALCULATING AN AMOUNT SAVED (FV)

Let us start by calculating the ending account balance (FV) when regular deposits (PMT) are made into an account. For this scenario, it is possible that there is an initial balance (PV) or no money in the account (PV = 0). Let's see an example where you start saving for your retirement and we'll examine both scenarios — when you have no money in the account at the start and when you have an initial balance.

EXAMPLE 5.1.1— SAVING FOR RETIREMENT

For the next ten years, you deposit $400 each month into a retirement savings account. You deposit the first payment one month from now. The account will pay 4.5%, compounded monthly. You start with $0 in the account at the start of the ten years. How much money will you have in the account at the end of ten years?

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	12	10×12=120	4.5	0	+400	**CPT** -60,479.23

Why is B/E set to END? This is because the first deposit is being made in one month, at the end of the first payment interval. When regular payments or deposits are made at the end of the payment interval, we call this an **ordinary annuity**. We do not need to do anything in the calculator (it is set to END by default). We will talk more about changing this setting in the later sections.

Why are P/Y and C/Y equal to 12? P/Y equals the number of payments or deposits per year. C/Y equals the number of times the interest rate compounds per year and the interest. Since both the compounding and deposits are monthly, both P/Y and C/Y equal 12.

Why does N equal 120? N equals 120 because you will make monthly deposits for 10 years. This will give you a total of 10×12 deposits.
Note: N = The total number of payments or deposits = Number of years × P/Y

Why does PV equal 0? PV is set to 0 because there is no money in the account at the start. PV is the initial (starting) balance in the account.

Finally, why does PMT equal +400? PMT equals to +400 because we make regular deposits of $400 into our savings account. We treat all deposits as positive because it is money going into a savings account[2].

Now we can calculate the amount saved at the end of 10 years (FV). Notice that the BAII Plus will give us a negative value for the future value. This minus sign (negative sign) indicates that the future value (FV) is going in the opposite direction of the deposits (PMT and PV). Don't include the minus sign (negative sign) in your final answer.

Conclusion: You will have $60,479.23 in your savings account at the end of 10 years.

Check Your Knowledge 5.1.1

Redo the above example but instead, start with a balance of $1,000 in the account. What will be the ending balance in the account? Drag the values that you would enter into the BAII Plus

Calculator Keys in the exercise below.

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Conclusion: You will have $62,046.22 in your savings account at the end of 10 years. Note that the initial deposit of $1,000 makes your ending balance $1,566.99 larger.

CALCULATING THE INTEREST EARNED

For all investments, loans, etc., we can consider the following formula to be true when calculating the interest earned or charged on our investment or loan:

Interest = Money Out – Money In = $ OUT – $ IN

In the case of regular deposits into a savings account, in this textbook, we will consider the initial deposit, PV (if there is one) to be "money in" since we are depositing this money INTO the account.[3] We consider the regular deposits, PMT, to be "money in" since they are also deposits going INTO the savings account. Finally, we consider FV to be "money out" since it is being withdrawn from the account when we close it:

PV	Interest	PMT	FV
Initial Deposit	**+ % Gain**	**+ Regular Deposits**	**= Final Withdrawal**
$ IN	$ IN	$ IN	$ OUT

This gives us the following equation for the interest earned:

$$\begin{aligned} \text{Interest Earned} &= \$\ \text{Out} - \$\ \text{IN} \\ &= \text{Final Withdrawal} - (\text{Initial Deposit} + \text{Regular Deposits}) \\ &= \text{FV} - (\text{PV} + \text{PMT} \times \text{N}) \end{aligned}$$

Check Your Knowledge 5.1.2

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KEY TAKEAWAYS

Key Takeaways: Regular Deposits into Savings Accounts

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YOUR OWN NOTES

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THE FOOTNOTES

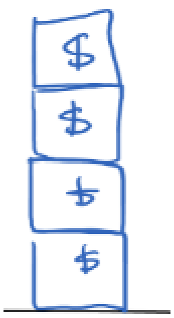
Notes

1. Other texts might treat PV and PMT as negative and FV as positive. What is most important is that we are aware that PV and PMT must have the same sign and FV must be opposite in sign.
2. Some texts might treat deposits into a savings account as negative. If they do, the initial deposit, PV (if non-zero) will also be negative and FV (the ending balance) will be positive.
3. Other texts or instructors might treat signs differently than we do here. What is most important is that we are aware that money flows in two different directions (in and out). It could be possible to consider the regular deposits as negative and the ending balance (FV) as positive. You will still get the correct answer because the deposits and final balance are opposite in sign.

5.2 Withdrawing from a Savings Account

Learning Outcomes

Calculate the payment size and interest earned for regular withdrawals from a savings account.



It is also possible to make regular withdrawals from a savings account. When you make these regular withdrawals, you're deducting from the money you've built up in the account. At the end of the annuity (when the account is closed), all remaining funds (FV) will need to be withdrawn. If nothing is specified for a remaining $ amount, assume it is zero.

PV	Interest	PMT	FV
Initial Deposit	**+ % Gain**	**= Regular Withdrawals**	**+ Final Withdrawal**
+	**+**	**−**	**0 or −**

A common example where these type of regular withdrawals are made is a retirement fund where the retiree withdraws a certain amount of money every month to pay their bills or an education account where a student withdraws money twice a year to pay their tuition.

See the sections below for key formulas, tips and examples related to calculations when withdrawing from a savings account.

CALCULATING THE PAYMENT SIZE

Let us start by calculating the size of your regular withdrawals (PMT) from an education fund. It is possible that there is an ending balance (FV) or nothing leftover at the end (FV = 0). We will examine both scenarios — when you have no money leftover and when you have an ending balance.

EXAMPLE 5.2.1

Your have $20,000 in your savings account to pay for your post-secondary education. You plan on making semi-annual withdrawals from the account for three years while you attend BCIT. The first withdrawal will be in six months. The savings account will earn 4%, compounded monthly. What will be the size of your semi-annual withdrawals?

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	2	12	3×2=6	4	+20,000	**CPT** −3,572.53	0

Why is B/E set to END? This is because the first withdrawal is being made in six months, at the end of the first payment interval. **Semi-annual payments** are payments that occur twice per year or once every six months. Again, we do not need to anything in the calculator (it is set to END by default).

Why does P/Y equal 2? The payments are semi-annual, or twice per year.

Why does C/Y equal 12? The account earns 4% compounded monthly (12 times per year).

Note that P/Y and C/Y are not equal in this example. When P/Y ≠ C/Y, we call this a **general annuity.**

Why does N equal 6? N equals six because you will make semi-annual withdrawals for 3 years. This will give you a total of 3×2 withdrawals. Remember that N = number of years × P/Y = total number of withdrawals.

Why does PV equal +20,000? The initial balance (or deposit) will be equal to PV and in this example, that initial balance equals $20,000. In this text, we will treat that initial balance (or deposit) as positive.

Finally, why does FV equal 0? Since all of the money will be withdrawn, FV = 0.

Now we can calculate the size of your withdrawals (PMT). Notice that the BAII Plus will give us a negative value for the payments. This minus sign (negative sign) indicates that the payments (PMT) are reducing the balance in the account until it eventually gets to zero (FV = 0). We will drop this minus sign (negative sign) for our final answer.

Conclusion: You will be able to withdraw $3,572.53 from your savings account every six months.

Check Your Knowledge 5.2.1

What if you wanted to have $3,000 remaining at the end of three years to travel after you graduate? Redo the above example but instead, have an ending balance of $3,000 remaining in the account. What will be the size of your regular withdrawals? Drag in the values that you would enter in your calculator.

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Conclusion: You will be able to withdraw $3,097.16 from your savings account every six months. Note that PMT and FV have the same sign (both negative). You can consider them both as withdrawals. We are withdrawing $3,097.16 every six months and withdrawing $3,000 from the account at the end of three years.

CALCULATING THE INTEREST EARNED

For all investments, again, interest is the difference between money in and money out.

$$\text{Interest} = \text{Money Out} - \text{Money In} = \$\text{ OUT} - \$\text{ IN}$$

In the case of annuities with regular withdrawals, we consider the initial deposit, PV, to be money in ($ IN) because this money is being deposited into the account. We consider the regular withdrawals, PMT, to be money out ($ OUT) because they are being withdrawn from the account. Finally, we consider FV (if there is any balance remaining at the end) to be money out ($ OUT) because we assume that the final amount (if it's not 0) will be withdrawn from the account when the account is closed.

PV	Interest	PMT	FV
Initial Deposit	**+ % Gain**	**= Regular Withdrawals**	**+ Final Withdrawal**
$ IN	$ IN	$ OUT	$ OUT

This gives us the following equation for the interest earned:

$$\begin{aligned}\text{Interest Earned} &= \$\ \text{OUT} - \$\ \text{IN}\\ &= (\text{Regular Withdrawals} + \text{Final Withdrawal}) - \text{Initial Deposit}\\ &= (\text{PMT} \times \text{N} + \text{FV}) - \text{PV}\end{aligned}$$

Check Your Knowledge 5.2.2

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KEY TAKEAWAYS

Key Takeaways: Regular Withdrawals from Saving Accounts

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YOUR OWN NOTES

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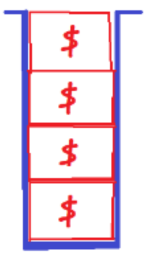
5.3 Loans and Down Payments

Learning Outcomes

Calculate the duration, selling price and cost of financing for loans with down payments.



The first type of debt annuity we will examine is the (**loan**) — an annuity where we borrow an initial amount of money (PV) and repay the loan with a series of equal-sized payments (PMT), at regular intervals, over the course of a fixed time period. At the end, we owe nothing (FV = 0).

PV	Interest	PMT	FV
Amount Borrowed	**+ %Charged**	**= Regular Payments**	**+ 0**
+	**+**	**–**	

Note: PV and PMT have opposite signs. To better understand this: PV is the initial amount we receive (loan amount) and PMT is the repayments of that loan that we must pay back after receiving the loan. The interest adds to the amount owed (we are charged interest each period on what we owe).

See the sections below for key formulas, tips and examples related to loans and down payments.

THE TIME REQUIRED TO REPAY A LOAN & KEY QUESTIONS

For loans, you can be asked to calculate the time, rate, initial amount borrowed or size of the loan payments. In the example below, we will calculate the amount of time required for Zhang Min to repay her line of credit that she took out to pay for school.

EXAMPLE 5.3.1

Today, Zhang Min borrowed $10,000 from her line of credit. Her line of credit charges 3.75%, compounded monthly. She can afford to pay $300 per month on her line of credit with the first payment one month from today. How many years will it take her to repay her line of credit?

Before we determine what to enter into your BAII Plus, let's ask a few important questions:

Key Questions: Loans

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This gives us the following values in the BAII Plus:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	12	**CPT** 35.255	3.75	+10,000	−300	0

Zhang Min will make 35 full-sized payments and a smaller final payment at the end of 36 months.
To calculate the number of years, we use the following formula:

$$\text{Number of Years} = \frac{\text{N}}{\text{P/Y}} = \frac{36}{12} = 3 \text{ years}$$

Conclusion: It will take 3 years for Zhang Min to repay her line of credit.

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DOWN PAYMENTS & KEY QUESTIONS FOR UNKNOWN SELLING PRICES

A **down payment** is a lump-sum payment made before you take out the loan. Down payments are often required when taking out a car loan or mortgage. The down payment will save money in interest charges over the duration of the loan. This is because it reduces the amount borrowed (PV):

$$\text{Amount Borrowed (PV)} = \text{Selling Price} - \text{Down Payment}$$

If we are asked to calculate the selling price when we know the value of the **down payment** and amount borrowed (PV), we can rework the above equation to solve for the selling price:

$$\text{Selling Price} = \text{Amount Borrowed (PV)} + \text{Down Payment}$$

EXAMPLE 5.3.2

Raj wants to buy an All Wheel Drive Tesla Model S. He can take out a 5-year loan with Tesla Lending. He must make a $10,000 down payment followed by monthly payments of $2,074/month with the first payment one month after the car is purchased. Tesla Lending charges Raj 3.75% effective on the loan. What is the selling price of the car?

Before we determine what to enter into your BAII Plus, let's ask a few important questions:

Key Questions: Loans with Known Down Payments and Unknown Selling Prices

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From the above answers and the values given in the problem, enter the following in the BAII Plus:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	1	5×12=60	3.75	**CPT** 113,484.44	−2074	0

Because PV equals to the amount borrowed, we know that Raj will borrow $113,484.44. We still need to calculate the selling price. In order to do this, we use the following formula:

$$\begin{aligned}\text{Selling Price} &= \text{Amount Borrowed (PV)} + \text{Down Payment} \\ &= \$113,484.44 + \$10,000 \\ &= \$123,484.44\end{aligned}$$

Conclusion: The selling price is $123,484.44 for the All Wheel Drive Tesla Model S.

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COST OF FINANCING ON LOANS & PRACTICE EXERCISE

We call the interest charged on loans the **cost of financing**. The same interest formula is used as before:

$$\text{Interest} = \text{Money Out} - \text{Money In} = \$\ \text{OUT} - \$\ \text{IN}$$

We consider the amount borrowed, PV, to be "money in" since we are receiving this money at the start of the loan. We consider the regular payments, PMT to be "money out." To calculate the total amount paid from the regular payments, calculate PMT×N since we will make N payments of size PMT. Finally, FV equals zero because nothing is owed at the end of a loan.

PV	Interest	PMT	FV
Amount Borrowed	**+ %Charged**	**= Regular Payments**	**+ 0**
$ IN	$ IN	$ OUT	

This gives us the following equation for cost of financing for loans:

$$\begin{aligned} \text{Interest Earned} &= \$\ \text{OUT} - \$\ \text{IN} \\ &= \text{Regular Payments} - \text{Amount Borrowed} \\ &= \text{PMT} \times \text{N} - \text{PV} \end{aligned}$$

Notice that the down payment is not used in the above formula. Only the loan payment size and amount borrowed are used to calculate the cost of financing for a loan.

Check Your Knowledge 5.3.2

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Check Your Knowledge 5.3.3

Raj feels like he can't afford the monthly payments for the 5-year loan (see Example 2). Instead, he takes out an 8-year loan. He will still make the $10,000 down payment followed by monthly payments with the first payment one month after the car is purchased. Tesla charges 4.25% effective on the 8-year car loan. How much extra interest will Raj pay if he takes out the 8-year loan instead of the 5-year loan (from Example 2)?

First we calculate the size of Raj's new payments. Drag in the values in the correct calculator keys:

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Because PMT = -1,392.25, we know that Raj will need to pay $1,392.25 per month if he takes out the 8-year loan. Use this payment size to calculate the amount of interest (cost of financing) on the 8-year loan:

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Conclusion: Raj will pay an extra $9,216 in interest if he takes out the 8-year loan instead of the 5-year loan.

YOUR OWN NOTES

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5.4 Leases and Annuities Due

Learning Outcomes

Calculate the rate and cost of financing for car leases and understand annuities due.

Leasing a car (or vehicle) is much like renting a vehicle. The **lessee** (person leasing the car) makes a series of equal-sized payments (**lease payments**), at regular intervals over the course of a fixed time period (**lease period**). These payments will be smaller than loan payments. This is because at the end of the lease, the lessee either must pay an additional amount called the **residual value** to purchase the car or they must return the car to the car dealership. To calculate any of the above values, we enter the following in for each value:

PV	Interest	PMT	FV
Amount Leased	**+ %Charged**	**= Lease Payments**	**+ Residual Value**
+	**+**	**–**	**–**

Leases are examples of **annuities due**. We will examine this term in the section below.

See the sections below for key formulas, tips and examples related to leases and annuities due.

ORDINARY ANNUITIES VS ANNUITIES DUE

When calculating an **ordinary annuity** (an annuity where the payments occur at the end of each payment interval), you do not likely need to change anything in your calculator. By default, your calculator will be set to **END** mode (which is the setting you need for ordinary annuities).

It is also possible that payments occur at the beginning of each interval. In this case, you need to "tell" your calculator that the payments are occurring at the beginning of the payment interval. You do this by setting the calculator to **BGN** mode. This type of annuity is called an annuity due.

HOW DO WE SWITCH BETWEEN BGN AND END ON THE BAII PLUS?

How Do We Set the BAII Plus to BGN?

- Hit **[2ND]** followed by **[PMT]** on your calculator (this gets you into the BGN/END menu).
- You will see END displayed on the screen. To change this to BGN, click on [**2ND**] and **[ENTER]**
- You will now see BGN displayed on the screen as well as in the top right of the screen.
- Hit [**2ND**] followed by **[CPT]** to exit the BGN/END menu.
- **Be careful!** Ordinary annuities are more common that annuities due so once you have completed your annuity due question, turn off BGN (ie: set your calculator back to END).

How Do We Set the BAII Plus to END?

- Hit [**2ND**] followed by [**PMT**] on your calculator.
- You will see BGN displayed on the screen. To change this to END, click on [**2ND**] and [**ENTER**].
- You will now see END displayed on the screen and you will see BGN disappear from the top right of the calculator screen.
- Hit [**2ND**] followed by [**CPT**] to exit the BGN/END menu.

Now, you can practice these steps in the exercise below.

Check Your Knowledge 5.4.1

Drag the keys onto the calculator screen in the correct order to switch from END to BGN.

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Drag the keys onto the calculator screen in the correct order to switch from BGN to END.

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Did you notice that the same keys are used to switch from END to BGN as BGN to END?

CALCULATING THE RATE CHARGED & DOWN PAYMENTS ON LEASES

When leasing a car, it can be required to make a down payment. If a down payment is made, we deduct that from the selling price to determine the amount leased:

$$\text{Amount Leased (PV)} = \text{Selling Price} - \text{Down Payment}$$

If we instead need the selling price, we re-shuffle the above equation to give:

$$\text{Selling Price} = \text{Amount Leased (PV)} + \text{Down Payment}$$

Let us now look at Raj's lease. He is hoping to lease a Tesla Model X.

EXAMPLE 5.4.1

Raj can lease a Tesla Model X car for three years. From the Tesla website, he finds the following pricing information:

- Selling Price: $110,880
- Term: 36 months
- Required Down Payment: $7,500
- Monthly payment size: $1,702, the first monthly payment is due the day the car is purchased
- Amount owed (residual value) in 3 years to purchase the Model X: $50,568.84

What effective interest rate is Tesla charging on the 3-year lease?

Check Your Knowledge 5.4.2

First, calculate the amount leased (PV):

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Next, enter the values into the BAII Plus:

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Conclusion: Tesla is charging 3.75% effective on their 3-year lease.

COST OF FINANCING FOR LEASES

We also call the interest charged on leases the **cost of financing**. The same interest formula is used:

$$\text{Cost of Financing} = \text{Money Out} - \text{Money In} = \$\ \text{OUT} - \$\ \text{IN}$$

In the case of car leases, we consider the selling price of the car, PV to be "money in." We consider the regular payments (PMT) and the residual value (FV) to be 'money out':

PV	Interest	PMT	FV
Amount Leased	**+ %Charged**	**= Lease Payments**	**+ Residual Value**
$ IN	$ IN	$ OUT	$ OUT

This gives us the following equation for cost of financing for leases:

$$\begin{aligned}\text{Cost of Financing} &= \$\text{ OUT} - \$\text{ IN}\\ &= (\text{Lease Payments} + \text{Residual Value}) - \text{Amount Leased}\\ &= (\text{PMT} \times \text{N} + \text{FV}) - \text{PV}\end{aligned}$$

Check Your Knowledge 5.4.3

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KEY TAKEAWAYS

Key Takeaways: Car Leases and Annuities Due

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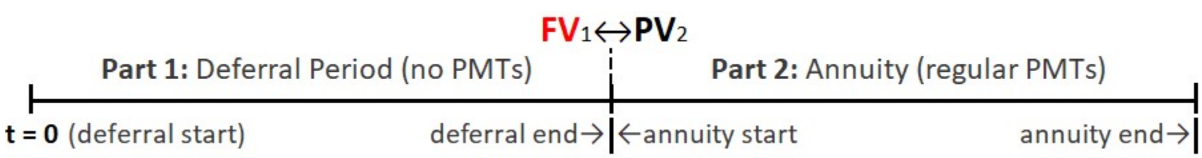
5.5 Deferred Annuities

Learning Outcomes

Calculate deferred withdrawals from a retirement fund and deferred payments on a debt.

To understand **deferred annuities**, let us first go back and examine the definition of an annuity. An **annuity** is “a series of equal-sized payments, at regular intervals, over a fixed period of time.” What then does it mean to defer this annuity? It means to delay (or defer) the regular payments for a period of time.

Because there is a deferral period and a payment (annuity) period, there are actually two parts to the problem:



Part 1: Deferral Period

- There are no payments in part 1 ($PMT_1 = 0$).
- The only money being added to the initial balance (PV_1) is the interest being earned (or charged).
- The ending balance from the deferral period (FV_1) equals the starting balance for the annuity (PV_2).

Part 2: Annuity with Regular Payments

- Payments are being made or withdrawn (PMT_2).
- Assume the final future value (FV_2) equals 0 unless told otherwise.

See the sections below for key formulas, tips and examples related to deferred annuities calculations.

EXAMPLES OF DEFERRED ANNUITIES

The most common example of a deferred annuity is a retirement fund where the investor is not yet ready to retire. They defer their withdrawals (payments) until they retire. In the mean time, the fund earns interest. The fund continues to earn interest as the investor withdraws money from the fund.

Other possible examples of deferred annuities are student loans[1] and in-store credit cards that offer "pay nothing for 18 months" promotions on their in-store credit cards. These are both less than perfect examples of deferred annuities in that they both come with exceptions.

In the case of the student loan, the student defers repaying the loan until after they have finished school[2]. What is special about this case is that the Government does not charge them any interest during the deferment. If the loan is granted by a bank or other financial institution, then the student gets charged interest during the deferral period.

In the case of "pay nothing for 18 months" promotion, the credit card holder defers payments (possibly waits for 18 months to start making payments) on the card. These cards and promotions come with a lot of fine print and exceptions. The credit card example becomes like a deferred annuity if the credit card holder fails to pay off the entire balance on the card before the 18 months is up. They get back-charged interest on the amount borrowed. If, after this happens, they set up a payment schedule to repay the amount owing on the card with regular, equal-sized payments, then the credit card example becomes a deferred annuity example.

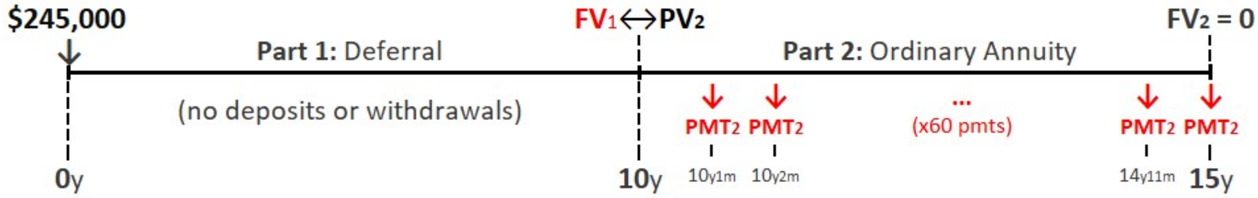
SAVING FOR RETIREMENT EXAMPLES

If you search the web for 'deferred annuity' — the savings deferred annuity related to planning for retirement will come up in at least the top 5 searches. It is the most common type of deferred annuity. When people plan for their retirement, they start saving before they retire. They usually do not need to withdrawn money from a retirement fund until they retire. In other words, they defer the withdrawals until retirement. Let us look at an example of this to better understand this type of deferred annuity.

Example 5.5.1: Deferred Withdrawals from a Retirement Fund with BGN Off

Ann and Sue sell their recreation property. They make $245,000 on the sale and deposit it immediately into a retirement fund. The fund earns 2.45%, compounded monthly, for the entire time. Exactly ten years after the sale, Ann retires. They now want to start supplementing their income with payments from the retirement fund. They want the payments to start one month after Ann retires. They want these payments to last for 5 years. How much can they withdraw per month from the retirement fund?

Let us first organize this information into a time diagram:



- The deferral period (part 1) occurs first.
- The deposit of $245,000 occurs at the very beginning of this deferral period.
- No payments nor withdrawals are made for the first 10 years (only interest is added for these 10 years).
- After the 10 years is complete, part 2, the annuity, begins.
- One month after the start of part 2 (annuity), the first withdrawal occurs.
- These withdrawals repeat for 5 years until no money is left in the annuity.

The tricky part with these types of problems with two parts is determining which part to start with. Start where the "known" money is. In this example, we know that Ann and Sue deposit $245,000 immediately ($PV_1$ = 245,000). For this reason, we will start with part 1 because PV_1 is known.

Part 1: Let us now organize the information for Part 1 into a BAII Plus table:

B/E	P/Y	C/Y	N_1	I/Y	PV_1	PMT_1	FV_1
--	12	12	10×12=120	2.45	+245,000	0	**CPT** -312,939.05

- Since there are no payments, B/E doesn't matter (that is why '—' is set for B/E).
- Because the interest rate is 2.45% compounding monthly, I/Y = 2.45 and C/Y = 12.
- Because there are no payments, match P/Y to C/Y (P/Y = 12).
- The deferral period lasts 10 years, so N_1 = 10×12=120.
- Remember, the $245,000 deposit occurs at the very beginning so PV_1 = +245,000.
- There are no payments for the deferral period so PMT_1 = 0.
- **CPT** FV_1 = −312,939.05. Ann and Sue's initial deposit grows to $312,939.05 after 10 years due to the interest earned on the deposit.
- The $312,939.05 ($FV_1$) will become the starting value for the annuity in part 2 (PV_2).

Part 2: Let us now organize the information for Part 2 into a BAII Plus table:

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_2
END	12	12	5×12=60	2.45	+312,939.05	**CPT** −5,546.95	0

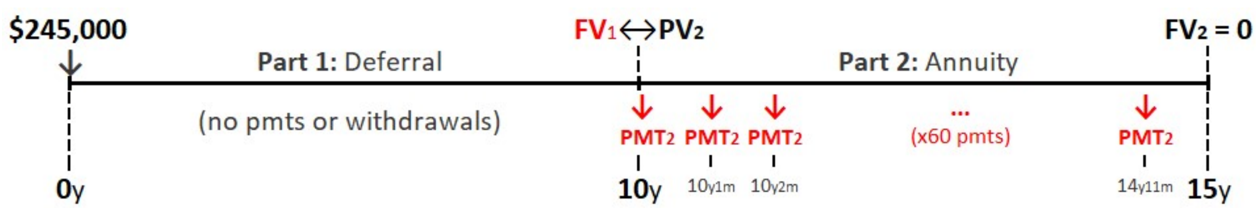
- The payments start one month after Ann retires (in one payment interval), so B/E = **END**.
- Because the interest rate is still 2.45% compounding monthly, I/Y = 2.45 and C/Y = 12.
- Ann and Sue receive monthly payments so P/Y = 12.
- They want these payments to last 5 years so, N_2 = 5×12=60.
- Use the (positive) value of FV_1 for PV_2. So, PV2 = +312,939.05.
- There is no money left at the end of 5 years so FV_2 = 0.
- Compute PMT_2 to get the size of Ann and Sue's monthly withdrawals.

Conclusion: Ann and Sue can withdraw $5,546.95 per month from their retirement fund.

Example 5.5.2: Deferred Withdrawals from a Retirement Fund with BGN On

What would change in Example 1 if Ann and Sue made the first withdrawal exactly 10 years after the sale of their property? Ie: what if their first withdrawal occurred the day that Ann retired?

Let us first look at the timeline for this problem:



- **Part 1** is the same as in Example 1, so we do not need to redo those calculations.
- **Part 2** has changed. The first withdrawal (PMT_2) now occurs exactly at the start (beginning) of that annuity.
- This means we set the calculator to **BGN** when calculating the annuity payments in part 2 (PMT_2):

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_2
END	12	12	5×12=60	2.45	+312,939.05	**CPT** −5,535.64	0

Conclusions:

- The amount withdrawn by Ann and Sue decreases slightly from $5,546.95 to $5,535.64 when they make the first withdrawal on the day that Ann retires.
- Withdrawing the first payment one month earlier means that Ann and Sue will miss out on the interest that would have been earned on that payment.
- This is because the money in the annuity continues to earn interest as the withdrawals are being made.
- As a result, Sue and Ann withdraw $11.31 less per month if they make the first withdrawal exactly 10 years after the sale of their property.

INTEREST EARNED WHEN SAVING FOR RETIREMENT

Again we use same interest formula:

$$\text{Interest Earned} = \text{Money Out} - \text{Money In} = \$\text{ OUT} - \$\text{ IN}$$

We need to be careful when calculating money in and money out for deferred annuities.

- What money was deposited in the account? Only the initial deposit (PV_1) is considered money in ($ IN).
- What money do we take out (withdraw)? Only the regular withdrawals (PMT_2) and final withdrawal (FV_2) in part 2 are considered to be money out ($ OUT).
- Because FV_1 does not get withdrawn but instead becomes the starting balance for the annuity (PV_2), it is not included in our interest calculation. The money is neither being deposited nor withdrawn.

PV_1	Interest_1	PMT_1	FV_1	PV_2	Interest_2	PMT_2	FV_2
Initial Deposit	**+ % Earned**	**+ $0**	**=Ending Balance Part 1**	**=Starting Balance Part 2**	**+ % Earned**	**= Regular Withdrawals**	**+ Final Withdrawal**
$ IN	$ IN	--	--	--	$ IN	$ OUT	$ OUT

This gives us the following equation for interest earned:

$$\begin{aligned} \text{Interest Earned} &= \$\text{ OUT} - \$\text{ IN} \\ &= (\text{Regular Withdrawals} + \text{Final Withdrawal}) - \text{Initial Deposit} \\ &= (\text{PMT}_2 \times \text{N}_2 + \text{FV}_2) - \text{PV}_1 \end{aligned}$$

Check Your Knowledge 5.5.0

Calculate the amount of interest Ann and Sue will earn in on their retirement savings plan from Example 2:

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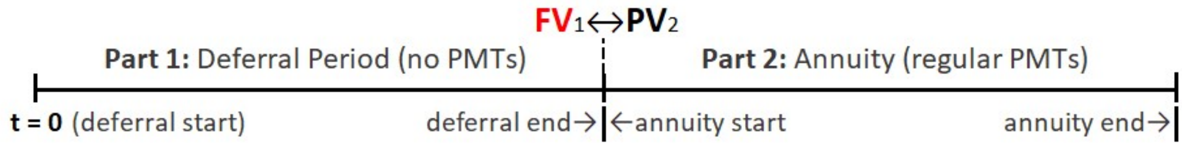
DEFERRING PAYMENTS ON A DEBT EXAMPLE & KEY TAKEAWAYS

It is quite common to defer payments when purchasing items, especially in retail. A retailer purchases their goods from a supplier. If cash flow is an issue for the retailer, they can often choose to defer the payments on the shipment until a later date. It is also common when consumers purchase items using “in-store” credit cards. Some stores have “Pay Nothing” promotions for their customers. If a customer purchases goods from the store on the in-store credit card, they can choose to make no payments on their purchase for a certain amount of time (6 months, 12 months, 18 months,…).

In the above cases, the retailer or customer owes the value of the amount of goods they purchased. They make no payments for a certain amount of time (they defer their payments).

After the deferral period has passed, they make payments to repay the value of the goods plus the interest charged during the deferral period. See the timeline describing this process:



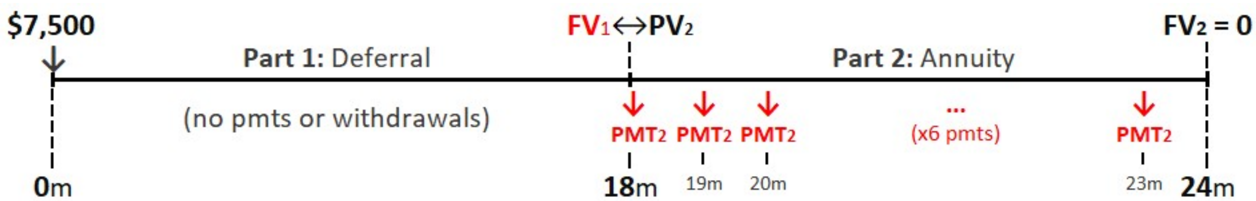
Part 1: The initial amount owing gathers interest. There are no payments in part 1 (the deferral period). The only money being added to the balance is the interest being charged. This problem is a compound interest problem (Chapter 4):

Part 2: Payments are now being made on the balance owing. This is the 'annuity' part of the problem.

Example 5.5.3: Deferred Repayment of Money Owed with BGN ON

Luis is renovating his kitchen. He takes advantage of a promotion offered by Home Depot: "Pay nothing for 18 months." He purchases $7,500 of materials on his Home Depot card. Home Depot charges him 28.8% compounded monthly. 18 months after the purchase, Luis makes his first of 6 monthly payments to pay off the credit card. What is the size of Luis's monthly payments?

Let us first look at the timeline for this problem:



From there, fill in the exercise below to figure out the size of Luis' payments.

Check Your Knowledge for Example 3

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To determine what to enter in the BAII Plus for Part 1, let's ask a few important questions:

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Next, enter the values for Part 1 into the BAII Plus:

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Conclusion: Luis will owe $11,493.71 at the end of the deferral period (end of the 18 months). This amount will become the starting balance (PV_2) in part 2.

Next, let's ask some key questions on what to enter in the BAII Plus for Part 2:

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Next, enter the values for Part 2 into the BAII Plus:

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Conclusion: Luis will need to make six monthly payments of $2,030.97 to pay off the credit card.

COST OF FINANCING WHEN DEFERRING PAYMENTS ON A DEBT

Again we use same interest formula:

$$\text{Cost of Financing} = \text{Money Out} - \text{Money In} = \$\ \text{OUT} - \$\ \text{IN}$$

We need to be careful when calculating money in and money out for deferred annuities.

- What is the money in ($ IN)? Only the initial amount borrowed (PV_1) is considered money in ($ IN).
- What is the money out ($ OUT)? Only the regular payments (PMT_2) are considered to be money out ($ OUT):
- You should note that because FV_1 is not a payment but instead becomes the starting balance for the annuity (PV_2), it is not included in our interest calculation.

PV_1	$Interest_1$	PMT_1	FV_1	PV_2	$Interest_2$	PMT_2	FV_2
Initial Deposit	**+ % Earned**	**+ $0**	**=Ending Balance Part 1**	**=Starting Balance Part 2**	**+ % Earned**	**= Regular Payments**	**+ $0**
$ IN	$ IN	--	--	--	$ IN	$ OUT	--

This gives us the following equation for cost of financing:

$$\begin{aligned}\text{Cost of Financing} &= \$\text{ OUT} - \$\text{ IN}\\ &= \text{Regular Payments} - \text{Amount Borrowed}\\ &= PMT_2 \times N_2 - PV_1\end{aligned}$$

Check Your Knowledge 5.5.3

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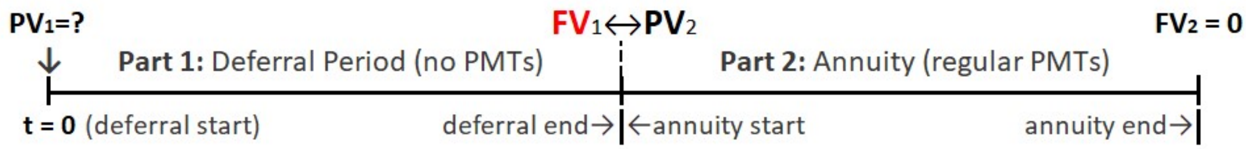
DEFERRED REPAYMENT WITH THE INITIAL AMOUNT OWED UNKNOWN

It is possible that the initial amount borrowed (PV_1) for a deferred annuity is unknown.

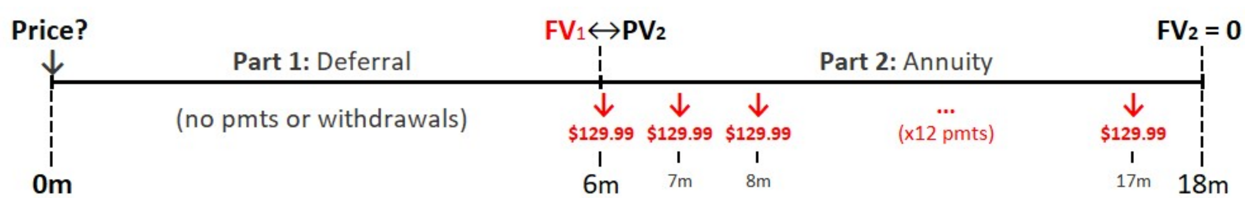


In the case, we must know the payment size in part 2 (PMT_2). For this reason, we start with part 2 (the annuity) in these cases. See Jessa's example below where she knows the size of the monthly payments required to finance a bedroom set at Ikea.

Example 5.5.4: Deferred Repayment with the Initial Amount Owed Unknown

Jessa is shopping at Ikea. She sees a sign for a beautiful bedroom furniture set. The sign reads the following. "Make no payments for 6 months followed by 12 easy payments of $129.99 for this entire bedroom set." Jessa has just moved into her new apartment and needs new bedroom furniture but she's a bit tight on cash right now. She is interested in the furniture and curious about how much the actual bedroom set costs. She reads the fine-print on the sign that states that Ikea charges an effective interest rate of 28.8% on financed purchases. What is the cost of financing that Jessa will pay if she purchases the bedroom set this way?

Let us first look at the timeline for this problem:



Check Your Knowledge 5.5.4

Let's find the price of the bedroom furniture set by breaking down this question into pieces. First, determine which "part" to start with:

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Next, drag and drop the correct values for Part 2 into the BAII Plus table:

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Jessa will owe $1,373.71 when she starts making regular payments. Use this amount (and make it negative) for FV1.

Next, drag and drop the correct values for Part 1 into the BAII Plus table:

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of the text. You can view it online here: https://pressbooks.bccampus.ca/businessmathematics/?p=1024#h5p-107

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The price of the furniture set is $1,191.51. Use this amount to calculate the cost of financing that Jessa will pay on her Ikea purchase:

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An interactive H5P element has been excluded from this version of the text. You can view it online here: https://pressbooks.bccampus.ca/businessmathematics/?p=1024#h5p-110

Conclusion: Jess will pay $368.37 in interest on her Ikea purchase.

YOUR OWN NOTES

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
- These notes are for you only (they will not be stored anywhere)
- Make sure to download them at the end to use as a reference

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THE FOOTNOTES

Notes

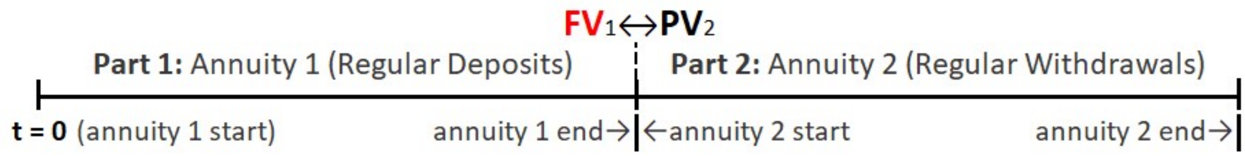
1. A student loan is a special case of a deferred annuity in that if the student receives a government-sponsored student loan, they do not need to pay any interest while they are in school.
2. If the Government of Canada has given out the student loan, then it is a special case of a deferred annuity. The student does indeed defer their payments until after they have finished school (they can wait up to 6 months after they are done school to start their payments).

5.6 Back to Back Annuities

Learning Outcomes

Calculate the initial deposit value or payment sizes in back-to-back annuities.

What are back-to-back annuities? They are a series of equal-sized, regular deposits (payments) over a fixed period of time (annuity 1) followed by a series of equal-size regular withdrawals for a fixed time period (annuity 2). In both cases, the balance in the account will be earning interest during the deposits and withdrawals. See the diagram below:



Notice that, unless there is a special deposit or withdrawal between the two annuities, the ending balance of the first annuity (FV_1) becomes the starting value of annuity 2 (PV_2) with one caution: we need to be careful of signs. FV_1 will be negative. PV_2 will be positive (they should be opposite in sign). We will talk more about this later during our first example.

See the sections below for key formulas, tips and examples related to back-to-back annuities calculations.

EXAMPLES OF BACK-TO-BACK ANNUITIES

It is common when saving for retirement, or for a child's education to save up by making regular deposits into an **RRSP** (registered retirement savings plan) or an **RESP** (registered education savings plan). Often, after making these regular deposits, the retiree (or student) starts making regular withdrawals from the account upon retirement (or upon starting school for the student).

THE SIGNS OF PV, PMT & FV FOR BACK-TO-BACK ANNUITIES

When calculating deposit or withdrawal amounts for back-to-back annuities, it is important to be careful of the signs of each of the values (for PV, PMT and FV). Let's examine the signs below:

Annuity 1: The initial balance (PV_1) is considered positive. This balance gathers interest. The subsequent payments (PMT_1) add to the existing balance in the account and are therefore also positive. At the end of the annuity, we consider the future value (FV_1) as the amount we would need to withdraw from the account to close the account. For this reason, the future value (FV_1) is recorded as negative:

PV_1	$Interest_1$	PMT_1	FV_1
Initial Deposit	**+ % Gain**	**+ Regular Deposits**	**= Ending Balance**
0 or +	**+**	**+**	**–**

Annuity 2: The initial balance (PV_2), in most cases, is the amount of money saved up in annuity 1 (FV_1). There is one exception to this rule – when there is a lump-sum deposited or withdrawn between the end of $Annuity_1$ and the start of $Annuity_2$. We will see examples of this later in this section.

PV_2	$Interest_2$	PMT_2	FV_2
Initial Deposit	**+ % Gain**	**= Regular Withdrawals**	**+ Final Withdrawal**
+	**+**	**–**	**0 or –**

For annuity 2, both the regular withdrawals (PMT_2) and the final ending balance (FV_2) deduct from the balance in the account. They should both be negative:

Putting this all together gives:

PV_1	$Interest_1$	PMT_1	FV_1	PV_2	$Interest_2$	PMT_2	FV_2
Initial Deposit	**+ % Earned**	**+ Regular Deposits**	**=Ending Balance Part 1**	**=Starting Balance Part 2**	**+ % Earned**	**= Regular Withdrawals**	**+ Final Withdrawal**
+	**+**	**+**	**–**	**+**	**+**	**–**	**–**

Notice that, we use the ending balance of annuity 1 (FV_1) becomes the starting balance of annuity 2 (PV_2) except we change the sign. FV_1 should be negative and PV_2 should be positive (they should be opposite in sign). We will talk more about this later during our first example.

DETERMINING THE SIZE OF THE REGULAR DEPOSITS (PMT_1)

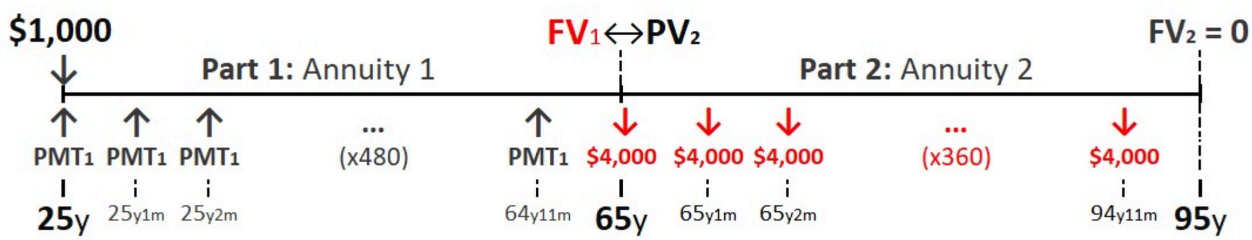
Some people plan for their retirement by deciding on the size of the withdrawals they would like to receive upon retirement (PMT_2). They then back-calculate the size of the deposits (PMT_1) they will need to make to achieve their retirement goals.

Let's have a look at Raj's retirement plan in the next example. Raj is very wise and starts saving when he turns 25!

Example 5.6.1

Today is Raj's 25^{th} birthday, and he has opened an account to start his retirement savings with an initial deposit of $1,000. He plans to make regular deposits into the account on a monthly basis, with the first deposit today. He estimated that, the retirement account will earn an average interest rate of 6% compounded annually. At age 65 he will turn his retirement saving into an annuity paying 4% compounded annually and he will be able to withdraw $4,000 per month for 30 years with the first withdrawal occurring on his 65^{th} birthday. How much does Raj's monthly deposit need to be in order to meet his retirement goals?

Let us first organize this information into a time diagram:



Next, we need to determine where to start. For back-to-back annuities, always start where the known payment is. In this example, we know the size of Raj's withdrawals during his retirement (PMT_2), therefore, we will "start" with part 2.

Let us now fill in the BAII Plus table for Part 2:

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_2
BGN	12	1	30×12=360	4	**CPT** +847,893.56	−4,000	0

- Because the withdrawals start right on Raj's 65th birthday, BGN is on.
- Raj's makes monthly withdrawals but the interest compounds annually so P/Y = 12 and C/Y = 1. Be careful when entering a different value into P/Y than C/Y in your BAII Plus.
- Raj wants to withdraw $4,000 per month for 30 years so N_2 = 30×12 and PMT_2 =

−4,000 (remember to make withdrawals negative).

- We assume there is nothing left in the account at the end of 30 years ($FV_2 = 0$) because we are not told otherwise.
- The present value (PV_2) becomes $847,893.56. This means that Raj will need $847,893.56 in his account when he retires in order to withdraw $4,000 per month for 30 years.
- The present value from the second annuity will become the future value for the first annuity but we change its sign
- Enter FV_1 as negative. Ie: $PV_2 = -FV_1$. This is because FV_1 is will be considered as the final withdrawal when ending annuity_1.

We can now calculate the size of Raj's monthly deposits (PMT_1). Let us fill in the BAII Plus table for Part 1:

B/E	P/Y	C/Y	N_1	I/Y	PV_1	PMT_1	FV_1
BGN	12	1	40×12=480	6	+1,000	**CPT** +436.95	−847,893.56

- Because the deposits start today, BGN is on for part 1.
- Again, Raj makes monthly payments and the interest is compounded annually, so P/Y = 12 and C/Y = 1.
- Because Raj makes monthly deposits for 40 years, $N_1 = 40 \times 12 = 480$.
- Finally, be careful of the signs for PV and FV. They must be opposite in sign.
- We will make PV_1 positive (as it is an initial deposit) and FV_1 negative (we treat it as a withdrawal at the end).

Conclusion: Raj needs to deposit $436.95 per month for the next 40 years to achieve his retirement goals.

INTEREST EARNED ON BACK-TO-BACK ANNUITIES

Again we use the usual interest formula:

$$\text{Interest Earned} = \text{Money Out} - \text{Money In} = \$\ \text{OUT} - \$\ \text{IN}$$

We need to be careful when calculating money in and money out for deferred annuities.

- All deposits are considered money in ($ IN).
- All withdrawals are both money out ($ OUT).
- Do not include FV_1 nor PV_2 in the $ IN or $ OUT calculations.

- Because FV_1 does not get withdrawn but instead becomes the starting balance for the annuity (PV_2), it is not considered money out.
- Similarly, because PV_2 does not get deposited but instead is actually the ending balance from annuity$_1$ (FV_1), it is not considered money in.

PV_1	Interest$_1$	PMT_1	FV_1	PV_2	Interest$_2$	PMT_2	FV_2
Initial Deposit	**+ % Earned**	**+ Regular Deposits**	**=Ending Balance Part 1**	**=Starting Balance Part 2**	**+ % Earned**	**= Regular Withdrawals**	**+ Final Withdrawal**
\$ IN	\$ IN	\$ IN	--	--	\$ IN	\$ OUT	\$ OUT

This gives us the following equation for interest earned:

$$\begin{aligned}\text{Interest Earned} &= \$\,\text{OUT} - \$\,\text{IN}\\ &= (\text{Regular Withdrawals} + \text{Final Withdrawal}) - (\text{Initial Deposit} + \text{Regular Deposits})\\ &= (\text{PMT}_2 \times \text{N}_2 + \text{FV}_2) - (\text{PV}_1 + \text{PMT}_1 \times \text{N}_1)\end{aligned}$$

Example 5.6.2

How much interest will Raj earn over the 70 years that his money is invested in Example 1a?

The Money Out, in this case, is the amount that Raj withdraws during his retirement:

$$\begin{aligned}\$\,\text{OUT} &= \text{Regular Withdrawals} + \text{Final Withdrawal}\\ &= \text{PMT}_2 \times \text{N}_2 + \text{FV}_2\\ &= \$4,000 \times 360 + 0\\ &= \$1,440,000\end{aligned}$$

The Money In is the amount Raj deposits into the account:

$$\begin{aligned}\$\,\text{IN} &= \text{Initial Deposit} + \text{Regular Deposits}\\ &= \text{PV}_1 + \text{PMT}_1 \times \text{N}_1\\ &= \$436.95 \times 480 + \$1,000\\ &= \$210,736\end{aligned}$$

Now take the difference between the money out and the money in (notice that neither FV_1 nor PV_2 are included in the \$ OUT nor \$ IN calculations):

$$\text{Interest Earned} = \$1,440,000 - \$210,736 = \$1,229,264$$

Conclusion: Raj will earn \$1,229,264 in interest over the 70 years that his money is invested!

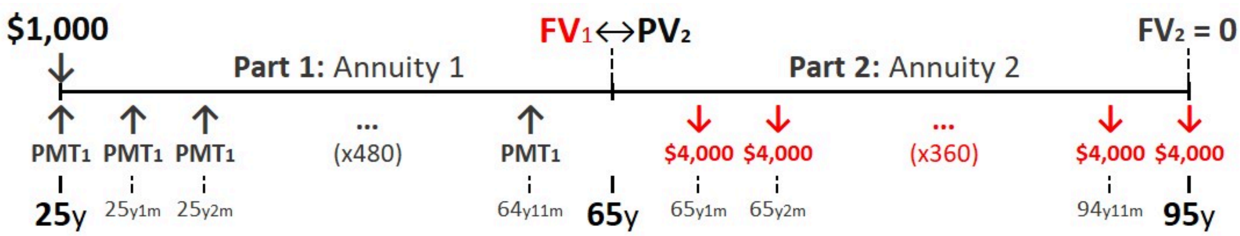
SWITCHING FROM BGN TO END

Let us examine Raj's retirement example (Example 1a) once again. In Example 1a, BGN was turned on for both annuity$_1$ and annuity$_2$. This was because Raj's first deposit was made immediately (at the start of annuity$_1$) and his first withdrawal was made exactly on this 65th birthday (at the start of annuity$_2$). Let us now look at an example where BGN is turned off (ie: the calculator is set to END).

Example 5.6.3

What would change if Raj withdrew his first retirement payment of \$4,000 (PMT$_2$) two months after his last deposit (last PMT$_1$)? How much would Raj need to deposit into the retirement fund each month (PMT$_1$) in this case?

Because Raj made his deposits into the saving account (annuity$_1$) at the beginning of each month then the last deposit would go into the account one month before his 65th birthday. His first withdrawal occurs two months after this last deposit. That means the first withdrawal occurs one month after his 65th birthday, which would be the end of the first payment interval for annuity$_2$. That means we set BGN to off (END) for annuity$_2$. Let us look at the new timeline for this question:



Notice that turning BGN 'off' (setting the calculator to END) will change the value of PV$_2$ for annuity$_2$. Let us start by re-calculating the value of PV$_2$:

B/E	P/Y	C/Y	N$_2$	I/Y	PV$_2$	PMT$_2$	FV$_2$
END	12	1	30×12=360	4	**CPT** +845,126.83	−4,000	0

Then, let's use the value for PV$_2$ in FV$_1$ but make FV$_1$ negative. Don't forget to turn BGN on for this calculation:

B/E	P/Y	C/Y	N$_1$	I/Y	PV$_1$	PMT$_1$	FV$_1$
BGN	12	1	40×12=480	6	+1,000	**CPT** +435.50	−845,126.83

Conclusion: Raj will need to deposit **$435.50** per month into his retirement fund.

Check Your Knowledge for Example 1c

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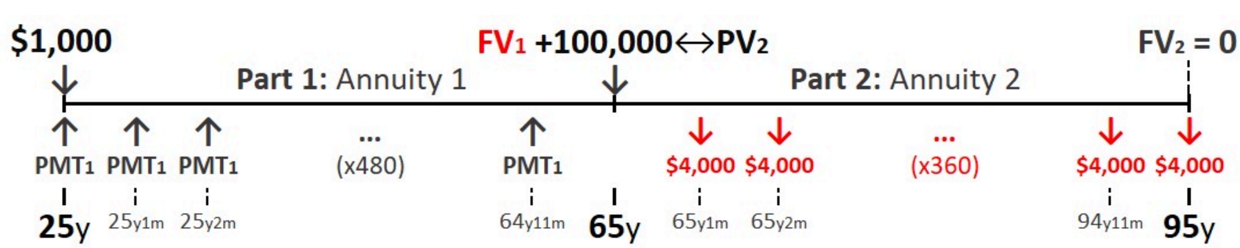
LUMP SUM DEPOSITS

It is possible to either deposit or withdrawal money between annuities. Let us again examine Raj's retirement example and see what happens if Raj deposits additional money into his retirement savings account when he turns 65.

Example 5.6.4

Raj anticipates downsizing (selling his house and buying a smaller property) when he turns 65. He thinks he can deposit $100,000 from the sale of his property into his retirement savings plan when he turns 65. How much are Raj's new monthly deposits into his savings plan (annuity 1) with this extra deposit into the retirement fund? Use the values from part c) and add the extra $100,000 deposit when Raj turns 65.

Let us first look at the timeline for this question:



Let us next examine the BAII Plus Table for Annuity$_2$.

Annuity 2 (Regular Withdrawals):

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_2
END	12	1	30×12=360	4	**CPT** +845,126.83	−4,000	0

Notice that nothing changes for Annuity$_2$ between examples 1c and 1d (PV_2 does not change). Where example 1d differs is in Annuity$_1$.

All together, Raj needs $845,126.83 saved up at the start of annuity$_2$ (PV_2). He receives an extra $100,000 from the sale of his property at that time. This means he only needs to save $745,126.83 during Annuity$_1$. This will be the value of FV_1:

$$\begin{aligned} \mathrm{PV}_2 &= \$845,126.83 \\ &= \$745,126.83 + \$100,000.00 \\ &= \mathrm{FV}_1 + \$100,000.00 \end{aligned}$$

We can see, above, that FV_1 must equal $745,126.83. Let us write out the formal equation to solve for $FV_1$when there is a lump-sum deposit between annuities:

$$\mathrm{FV}_1 = \mathrm{PV}_2 - \text{Lump Sum Payment}$$

Now that we know FV_1, we can calculate the new value for PMT_1.

Annuity 1 (Regular Deposits)

B/E	P/Y	C/Y	N_1	I/Y	PV_1	PMT_1	FV_1
BGN	12	1	40×12=480	6	+1,000	**CPT** +383.34	−745,126.83

Conclusion: Raj will need to deposit **$383.34** per month into his retirement fund.

Check Your Knowledge for Example 1d

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INTEREST EARNED WITH LUMP SUM DEPOSITS

When calculating interest earned on back-to-back annuity problems with lump-sum payments, there is an additional value we include in the money in ($ IN) — the lump sum deposit:

$$\begin{aligned}\$\text{ IN} &= \text{Initial Deposit} + \text{Regular Deposits} + \text{Lump Sum Deposit} \\ &= \text{PV}_1 + \text{PMT}_1 \times \text{N}_1 + \text{Lump Sum Deposit}\end{aligned}$$

The money out ($ OUT) does not change. Taking the difference between the money out and money in gives the following equation for interest earned on back-to-back annuities with lump-sum deposits:

$$\text{Interest Earned} = (\text{PMT}_2 \times \text{N}_2 + \text{FV}_2) - (\text{PV}_1 + \text{PMT}_1 \times \text{N}_1 + \text{Lump Sum Deposit})$$

Let us now look at Raj's retirement example again to see an example of this calculation.

Example 5.6.5

How much interest will Raj earn in over the 70 years of his retirement investment in Example 1d?

The $ IN will now include the additional $100,000 deposit as well as the initial deposit (PV_1) and the regular monthly deposits (PMT_1) into the savings account (annuity_1):

$$\$\text{ IN} = \$1,000 + 480 \times \$383.34 + \$100,000 = \$285,003.20$$

The $ OUT will not change:

$$\$\text{ OUT} = 360 \times \$4,000 = \$1,440,000$$

Taking the difference between $ OUT and $ IN gives:

$$\text{Interest} = \$1,440,000 – \$285,003.20 = \$1,154,996.80$$

Conclusion: Raj will earn $1,154,996.80 in interest over the 70 years that his money is invested.

Check Your Knowledge for Example 1e

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REGISTERED EDUCATION SAVINGS PLAN (RESP)

It is common when saving for a child's education to save up by making regular deposits into an RESP (registered education savings plan). There is a grant called the **Canada Education Savings Grant (CESG)** that adds 20% to these regular deposits (up to a maximum of $500 per year). Let us examine how this grant works for Sofia's RESP in the example below.

Example 5.6.6

Dmitry and Elena just had a baby girl, Sofia. They set up an **RESP** for Sofia. Given below are the terms:

- They deposit $625 every quarter into the RESP
- The first deposit occurs 3 months after Sofia is born
- The last deposit occurs on Sofia's 18^{th} birthday
- The deposits receive the CESG grant — 20% is added to each deposit by the government

- Sofia makes her first of 8 semi-annual withdrawals on her 18th birthday to attend university
- The education fund earns 4.25% compounded monthly for the entire time

What is the size of Sofia's semi-annual withdrawals?

Before we determine what to enter on the time diagram & BAII Plus, let's ask a few important questions:

Key Questions to Get Started on Example 2

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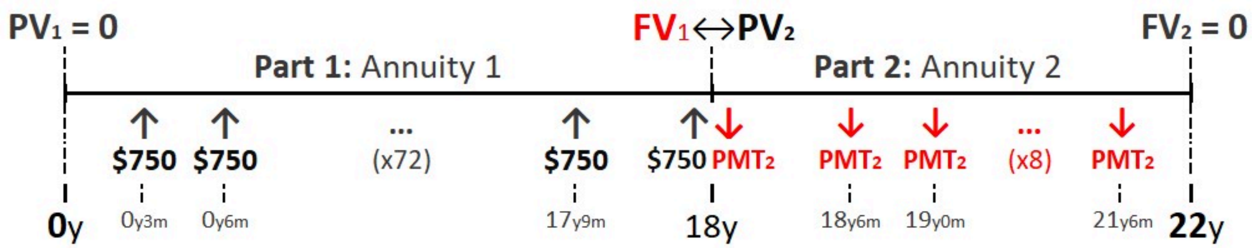
Let us gather all of this information into a timeline:



Always start where the known payment (PMT) is. We know that \$750 per quarter will be deposited into the RESP (= PMT_1). Therefore, "start" with Part 1.

Annuity 1 (Regular Deposits):

B/E	P/Y	C/Y	N_1	I/Y	PV_1	PMT_1	FV_1
END	4	12	18×4=72	4.25	0	+750	**CPT** −80,614.76

Enter the value for FV1 into PV_2. Don't forget to make PV_2 positive. We can now calculate PMT_2:

Annuity 2 (Regular Withdrawals):

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_2
BGN	2	12	4×2=8	4.25	+80,614.76	**CPT** −10,840.65	0

Conclusion: Sofia can withdraw \$10,840.65 every 6 months to pay for university [1].

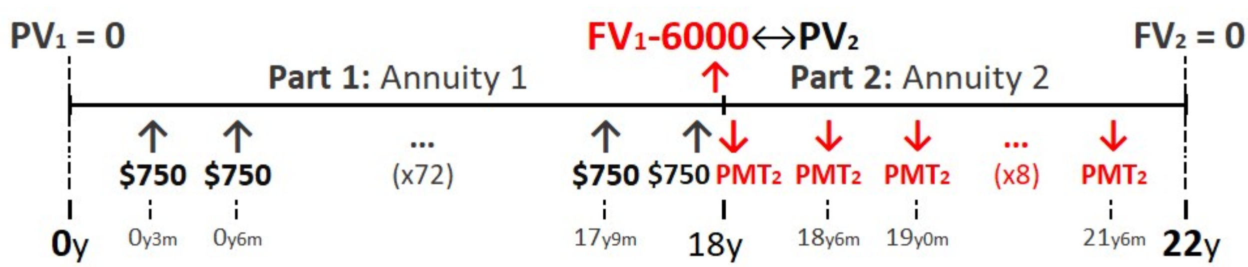
LUMP SUM WITHDRAWALS

Let us now examine the scenario where a lump-sum withdrawal occurs between annuities. We will see what happens if Sofia makes a lump-sum withdrawal on her 18th birthday in the example below:

Example 5.6.7

Redo Example 2 but add the following: on Sofia's 18th birthday, she purchases a car for \$6,000. Because she will be using the car to get to and from university, she is able to withdraw \$6,000 from her RESP fund. What will be the size of her new semi-annual withdrawals if everything else remains the same on Sofia's RESP account?

With the new \$6,000 withdrawal when Sofia turns 18, the timeline will now become:



The value of FV_1 will remain the same:

Annuity 1 (Regular Deposits):

B/E	P/Y	C/Y	N_1	I/Y	PV_1	PMT_1	FV_1
END	4	12	18×4=72	4.25	0	+750	**CPT** −80,614.76

Now, let's use the value of FV1 to calculate PV_2.

$$\begin{aligned} PV_2 &= FV_1 - \$6,000 \\ &= \$80,614.76 - \$6,000 \\ &= \$74,614.76 \end{aligned}$$

We can now enter $74,614.76 in for PV_2 and calculate the new value for PMT_2.

Annuity 2 (Regular Withdrawals):

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_2
BGN	2	12	4×2=8	4.25	+74,614.76	**CPT** −10,033.80	0

Conclusion: Sofia can now withdraw $10,033.80 every 6 months from her RESP fund.

INTEREST EARNED WITH LUMP SUM WITHDRAWALS

When calculating interest earned on back-to-back annuity problems with lump-sum withdrawals, there is an additional value we include in the money out ($ OUT) — the lump sum withdrawal:

$$\begin{aligned} \$\ \text{OUT} &= \text{Lump Sum Withdrawal} + \text{Regular Withdrawals} + \text{Final Withdrawal} \\ &= \text{Lump Sum Withdrawal} + \text{PMT}_2 \times \text{N}_2 + \text{FV}_2 \end{aligned}$$

The money in ($ IN) does not change. Taking the difference between the money out and money in gives the following equation for interest earned:

$$\text{Interest Earned} = (\text{Lump Sum Withdrawal} + \text{PMT}_2 \times \text{N}_2 + \text{FV}_2) - (\text{PV}_1 + \text{PMT}_1 \times \text{N}_1)$$

Let us now look at Sofia's RESP example again to see an example of this calculation.

Example 5.6.8

How much interest does Sofia's RESP account earn in Example 2b over the 22 years?

Let us first calculate the Money In to the RESP account over the 22 years:

$$\text{Money In} = 72 \times \$750 = \$54,000$$

The Money Out will include the \$6,000 withdrawal:

$$\text{Money Out} = \$6,000 + 8 \times \$10,033.80 = \$86,270.40$$

Taking the difference between the Money Out and In gives:

$$\text{Interest} = \$86,270.40 – \$54,000 = \$32,270.40$$

Sofia's RESP will earn \$32,270.40 in interest over the 22 years that the money is invested.

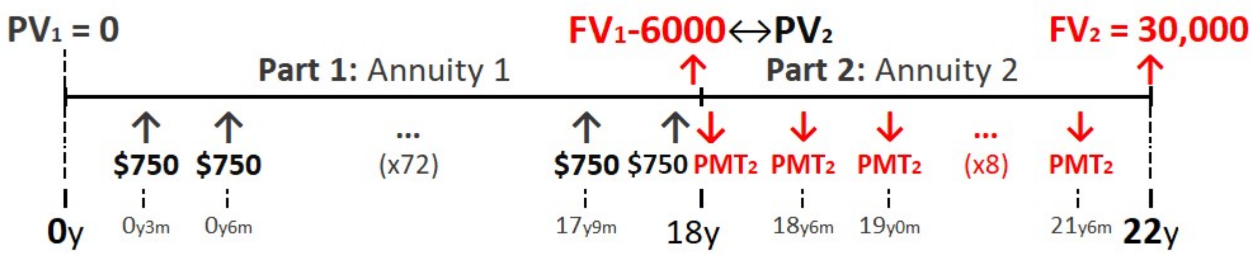
FINAL WITHDRAWALS

Let us finish this (very long) section with one final topic — money left over at the end. If there is any money over at the end of a back-to-back annuity, this amount will be entered into FV_2 and we make it negative (it is treated as a final withdrawal). Let us look again at Sofia's RESP example and assume Sofia wants money left over at the end.

Example 5.6.9

Redo Example 2b but assume Sofia wants \$30,000 left in her RESP after she graduates from university to help pay for her MBA. Calculate the size of her semi-annual withdrawals in this case.

With the \$30,000 left at the end, the timeline now becomes:



All of Annuity_1 will remain the same as well as PV_2 (the same as in Example 2b). We will start our calculations with Annuity_2. Almost all of the values are the same except FV_2: enter the ending balance amount (\$30,000) into FV_2 and make it negative, Then calculate Sofia's new withdrawal size (PMT_2):

Annuity 2 (Regular Withdrawals):

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_2
BGN	2	12	4×2=8	4.25	+74,614.76	**CPT** −6,629.23	−30,000

Conclusion: Sofia can withdraw $6,629.33 every 6 months and still have $30,000 left over at the end.

YOUR OWN NOTES

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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THE FOOTNOTES

Notes

1. The current maximum allowable withdrawal amount on RESP's is $5,000 in the first quarter while attending post-secondary. Let's assume that this maximum allowable amount will increase in 18 years once Sofia attends university.

5.7 Bonds

Learning Outcomes

Calculate the coupon payment size, market value or gain/loss for bonds.

If a corporation or government is looking to raise money (capital), they can issue bonds. The **issuer** (the company or government) must pay the **bond holder** (owner of the bond) a series of equal-sized regular interest payments for a fixed period of time. The size of these payments is determined by the agreed-upon interest rate (**coupon rate**). At the end of the fixed period of time, or **maturity date,** the issuer must repay the bond holder the **principal** (the amount of money the bond issuer borrowed).

We consider the bond is a debt owed by the issuer to the bond holder. The amount owed never increases because the issuer pays the interest owed each period to the holder in the form of a **coupon payment**. This is why the final amount owed by the issuer to the bond holder (the face value) will be the principal (amount borrowed).

There are several different types of calculations for bonds — see the sections below for the key formulas, tips and examples related to bond calculations.

CALCULATING COUPON PAYMENTS FOR BONDS

The coupon payment (PMT) is the interest earned in one period (6 months):

$$PMT = \text{Principal} \times i$$

where the principal is the initial purchase price of the bond (when the bond is purchased from the issuer) and i is the **periodic rate**. The periodic rate is the interest rate for one period.

Example 5.7.1

Amir purchased a $3,000 bond that has a 10-year term (will mature in 10 years). The bond has a coupon rate of 5%, compounded semi-annually. What is the size of the semi-annual coupon payment?

Check your Knowledge for Example 1

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CALCULATING THE FAIR MARKET VALUE OF A BOND

Often, the bond holder can sell the bond (transfer the bond) to a **secondary bond holder** before the end of the **term**. Because interest rates change throughout the term of the bond, the bond can be worth different amounts at different times throughout its term. The amount the purchaser (secondary bond holder) is willing to pay is called the **fair market value of the bond**. The fair market value is determined by the current interest rate, the size of the coupon payment, the number of years remaining in the term (number of years before the maturity date) and the **face value** of the bond.

We will use the BAII Plus to calculate the fair market value of a bond:

PV	Interest	PMT	FV
Fair Market Value	+ % Market Rate	= Coupon Payments	+ Face Value
–	–	+	+

Bonds are the ONLY type of investment in this text where we will have a negative value for PV and a positive value for FV. To make sense of this, imagine the bond from the perspective of the bond holder. The bond holder initially ‘lends’ money to the bond issuer when purchasing a bond from the issuer. We consider this amount lent (PV) to be negative because this is the amount the bond holder must outlay to purchase the bond.

Each period, the issuer will owe the bond holder interest on that loan. The issuer pays this interest (coupon payment) to the holder each period. The issuer also repays the face value of the bond (FV) at the end to the holder[1]. For this reason, we consider PMT and FV as positive because they are money that the bond holder receives.

Example 5.7.2

Francis purchases a $4,000 bond that has a 30-year term. The bond has a coupon rate of 5.25%, compounded semi-annually and a semi-annual coupon payment of $65. If Francis chooses to sell the bond after 10 years when the current interest rate is 3%, compounded semi-annually, what is the fair market value of the bond?

Check your Knowledge for Example 2

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Conclusion: Francis can sell the bond for $4,149.58 in 10 years.

CALCULATING THE GAIN OR LOSS WHEN SELLING A BOND

If the interest rate (coupon rate) on a bond is above the market rate then the bond increases in value **(sells at a premium)**. If the interest rate (coupon rate) on a bond is below the market rate, then the bond goes down in value **(sells at a discount)**. The gain (or loss) is calculated by:

$$\text{Gain (or Loss)} = \text{Selling Price} - \text{Purchase Price}$$

Let us use this formula to determine Francis' gain or loss in Example 2.

Example 5.7.3: Selling at a Premium

What is Francis' gain or loss on the sale of his bond in Example 2?

Check your Knowledge for Example 2b

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Conclusion: Francis will make a $149.58 gain when he sells the bond in 10 years.

Example 5.7.4 — Selling at a Discount

Redo Example 2 with an interest rate (market rate) of 6% when Francis chooses to sell the bond in 10 years. What is Francis' gain or loss on the sale of the bond?

Check your Knowledge for Example 2c

First, we need to calculate the fair market value of the bond in 10 years:

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of the text. You can view it online here: [*https://pressbooks.bccampus.ca/businessmathematics/?p=998#h5p-68*](https://pressbooks.bccampus.ca/businessmathematics/?p=998#h5p-68)

The fair market value of the bond will be $2,728.69 when Francis sells the bond in 10 years. We can use this value to now calculate Francis' gain or loss:

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Conclusion: Francis will lose $1,271.31 on the sale of the bond.

KEY TAKEAWAYS FOR BONDS

Key Takeaways for Bonds

- Coupon Payment (PMT) = PV × i = Principal × Periodic Rate

- Periodic Rate = Nominal Rate ÷ 2 (Bonds are always semi-annual)
- For bonds, PV will be negative and PMT and FV are positive.
- Bonds are the ONLY type of investment in this text where PV will be negative.
- Bonds are the ONLY type of investment in this text where FV will be positive.
- If the market rate rises above the coupon rate, the bond decreases in value.
- If the market rate drops below the coupon rate, the bond increases in value.
- The gain or loss on the sale of bonds is given by:
  $\text{Gain (or Loss)} = \text{Selling Price} - \text{Purchase Price}$

YOUR OWN NOTES

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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THE FOOTNOTES

Notes

1. Information thanks to https://www.investopedia.com/terms/b/bond.asp ; https://en.wikipedia.org/wiki/Bond_(finance) and https://www.investopedia.com/articles/investing/062813/why-companies-issue-bonds.asp

5.8 Perpetuities

Learning Outcomes

Calculate the payment sizes or present values for regular and deferred perpetuities.

A **perpetuity** is like a bond, but with no fixed term (no fixed **maturity date**). If a corporation issues a perpetuity to an investor, the perpetuity will continue making payments to this investor indefinitely[1]. Common examples of perpetuities are scholarship funds, perpetual trusts and war bonds, like the British Treasury Bonds issued for World War 1[2]. Finally, some business investments can also be treated like perpetuities[3].

There are several different types of perpetuities — see the sections below for the key formulas, tips and examples related to perpetuity calculations.

PAYMENTS FOR ORDINARY PERPETUITIES

If the payments for a perpetuity are withdrawn at the end of each interval, we call this an **ordinary perpetuity**. The payment (PMT) for an ordinary perpetuity can be given by the following formula:

$$\mathrm{PMT} = \mathrm{PV} \times i$$

where PV is the initial value of the perpetuity and i is the periodic rate (the rate per period).

We can also use the BAII Plus to calculate the PMT. There are a few 'tricks' to know when using the BAII Plus:

- B/E = "END" for ordinary perpetuities
- N = 1000 × P/Y. Use total # years = 1,000 for perpetuities[4].
- PV = the initial balance of the perpetuity.
- FV = 0. The funds are never withdrawn. The amount withdrawn equals 0 for this reason.
- **CPT** PMT. Calculate the value of the payment (PMT).

Example 5.8.1

Sasha, a wealthy BCIT Alumnus, just donated $100,000 to create a scholarship at BCIT for first year business students. The scholarship will be awarded semi-annually and the fund will earn 5% compounded semi-annually. The first scholarship will be awarded in 6 months. What will be the size of the semi-annual scholarships awarded?

Using the formula:

$$\begin{aligned}\text{PMT} &= \text{PV} \times i = \$100,000 \times \frac{0.05}{2} \\ &= \$100,000 \times 0.025 = \$2,500\end{aligned}$$

Using the BAII Plus:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	2	2	1000×2=2000	5	+100,000	**CPT** −2,500	0

Conclusion: BCIT can pay out a scholarship of $2,500 every 6 months.

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THE PRESENT VALUE FOR ORDINARY PERPETUITIES

There are also two ways to calculate the present value of an ordinary annuity. We can use the following formula:

$$\text{PV} = \frac{\text{PMT}}{i}$$

or we can, again, use the BAII Plus and use the same tricks as when calculating the payment (PMT).

Let us look again at Sasha's example but instead, let us figure out the amount Sasha needs to donate.

Example 5.8.2

How much does Sasha (the wealthy BCIT alumnus) need to donate if he wants BCIT to give out $3,000 semi-annual scholarships with the first scholarship awarded in 6 months. Assume the fund still earns 5% compounded semi-annually.

Using the formula:

$$\mathrm{PV} = \frac{\mathrm{PMT}}{i} = \frac{\$3,000}{0.025} = \$120,000$$

Using the BAII Plus:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	2	2	1000×2=2000	5	**CPT** +120,000	−3,000	0

Conclusion: Sasha will need to donate $120,000 to the scholarship fund at BCIT.

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https://pressbooks.bccampus.ca/businessmathematics/?p=1010#h5p-72

THE PRESENT VALUE FOR PERPETUITUES DUE

It is possible to have a perpetuity that gives out the first payment immediately. We call this a **perpetuity due**. An example of this is a scholarship fund where the first scholarship is awarded as soon as the fund has been created. [5]

If payments are withdrawn from a perpetuity fund at the start of each payment interval (perpetuity due), there will need to be slightly more in the perpetuity fund to account for the initial balance in the account dropping in value right at the start of the perpetuity. That “slightly more” amount will be the value of the payment[6]:

$$\mathrm{PV}_{due} = \frac{PMT}{i} + PMT$$

Let us now revisit Sasha’s scholarship example and determine the size of Sasha’s donation if the scholarship fund is a perpetuity due.

Example 5.8.3

Suppose that Sasha would like to donate enough money such that the semi-annual scholarships are still $3,000 but the first scholarship will be awarded immediately. Assume the scholarship fund still earns 5% compounded semi-annually. How much more will Sasha need to donate?

Using the formula and i = 0.025 and PMT = 3,000 gives:

$$PV_{due} = \frac{\$3,000}{0.025} + \$3,000 = \$123,000$$

Using the BAII Calculator gives:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
BGN	2	2	1000×2=2000	5	**CPT** +123,000	−3,000	0

Sasha will need to donate $123,000 if the first scholarship is awarded immediately. Taking the difference between the required donation amount donated for Example 2 (an ordinary perpetuity) and Example 3 (a perpetuity due) gives:

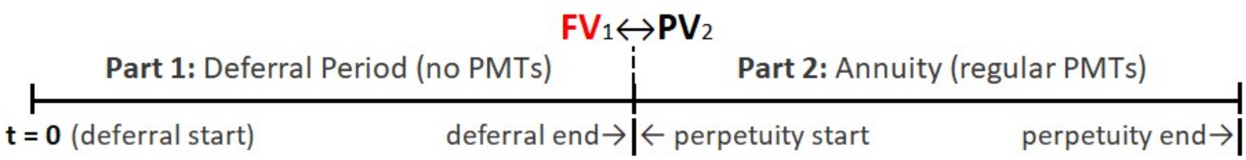
$$\text{Difference} = \$123,000 - \$120,000 = \$3,000$$

Conclusion: Sasha will need to donate $3,000 more if the first scholarship is awarded today instead of 6 months from now. It is no coincidence that the difference is equal to the payment size.

THE PRESENT VALUE FOR DEFERRED PERPETUITUES

It is possible to defer the payments received from a perpetuity. A first example is if a non-profit or charity receives a donation to cover future costs for the organization[7]. Another example of a perpetuity is a business decision where there is an initial amount invested to start the business (PV_1) and then it takes several years for the business to start earning income[8].

When calculating payment sizes for deferred perpetuities or the initial amount needed to be deposited at the start of the perpetuity, it is important to remember that there are actually two parts to the deferred perpetuity problem:



Part 1: The initial balance gathers interest. There are no payments nor withdrawals in part 1

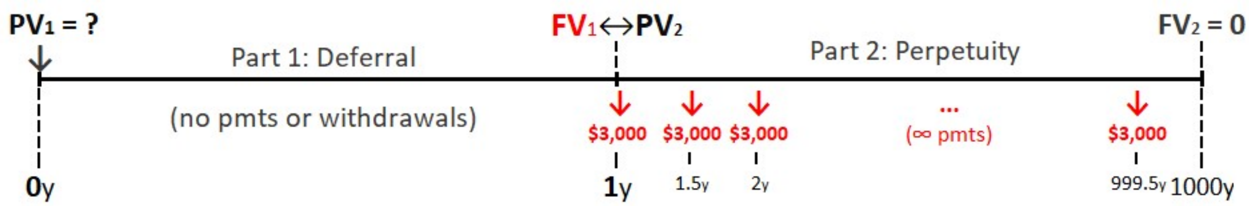
(the deferral period). The only money being added to the balance is the interest being earned (or charged). This problem is a compound interest problem (Chapter 4):

Part 2: Payments are now being withdrawn. These payments exactly equal to the interest earned on the current balance in the perpetuity account. The starting balance for the perpetuity (PV_2) equals to the ending balance from the deferral period (FV_1) .

Example 5.8.4

What if Sasha's scholarship fund (from Examples 1 to 3) doesn't give out its first scholarship for one year? How much would Sasha need to donate if the semi-annual scholarships are still $3,000 and the fund still earns 5% compounded semi-annually?

Let us first draw out the timeline for this problem:



Because we know the payments for part 2, start there. We can either use the formula or the BAII Plus to do the calculations for this problem. Let us start by using the BAII Plus:

Part 2: Perpetuity (using the BAII Plus)

Let us 'start' part 2 exactly when the first scholarship is awarded. For this reason, B/E will be set to BGN:

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_2
BGN	2	2	1000×2=2000	5	**CPT** +123,000	−3,000	0

Notice that PV_2 equals to the amount Sasha would need to donate in Example 3. We will now enter that value into FV_1 and calculate the amount donated initially (PV_1).

Part 1: Deferral Period (using the BAII Plus)

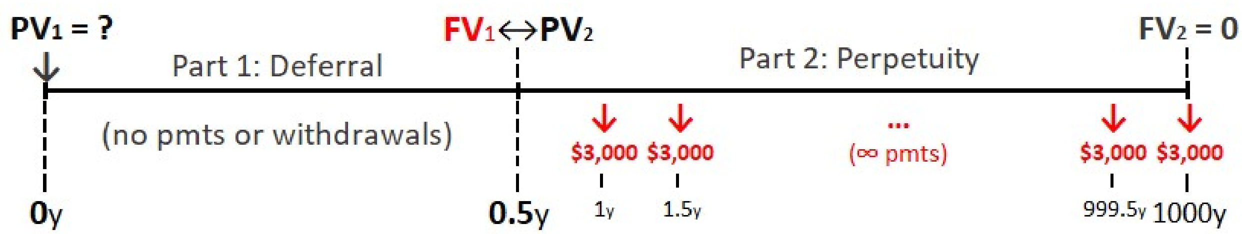
There are no payments nor withdrawals for part 1 so enter '—' for B/E. Also, because the deferral period lasts for 1 year, $N_1 = 1\times2 = 2$.

B/E	P/Y	C/Y	N_1	I/Y	PV_1	PMT_1	FV_1
––	2	2	1×2=2	5	**CPT** +117,073.17	0	−123,000

Conclusion: Sasha will only need to donate $117,073.17 if the first scholarship is awarded in one year.

Method 2 — Using Formulas:

We will again start with part 2. Let us rewrite the timeline, slightly differently, however:



Notice that we will only run the deferral period for 6 months and start the perpetuity (part 2) 6 months before the first scholarship is withdrawn. This makes the calculation easier for part 2:

Part 2: Perpetuity (using Formulas)

We now have 'started' part 2 one payment period earlier. This now makes the perpetuity in part 2 an ordinary annuity. We use the following formula to calculate its present value:

$$\mathrm{PV}_2 = \frac{\mathrm{PMT}_2}{i} = \frac{\$3,000}{0.025} = \$120,000$$

There will need to be $120,000 in the scholarship fund in 6 months. We can find the present value of this amount using the compound interest formula from Chapter 4.

Part 1: Deferral Period (using Formulas)

$$\mathrm{PV}_1 = \frac{\mathrm{FV}_1}{(1+i)^n} = \frac{\$120,000}{(1+0.025)^1} = \$117,073.17$$

Conclusion: Again, we find that Sasha will need to donate $117,073.17 now if the first scholarship is awarded in 1 year.

PAYMENTS FOR DEFERRED PERPETUITIES

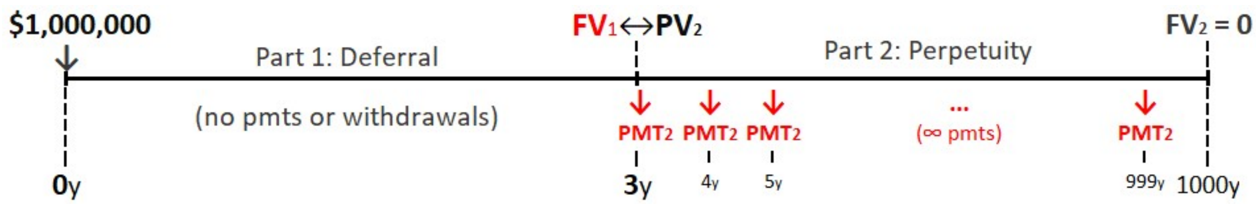
It is possible to use formulas to calculate the payment size for deferred perpetuities. We will however, just use the BAII Plus method for this section.

Let us examine a similar donation example. In this example, Ezra, a wealthy philanthropist will make a large donation to help out a medical center that will not open for several years.

Example 5.8.5

Ezra, a wealthy philanthropist, donates $1,000,000 to the Moshi Medical Diagnostics Center, located in Moshi, Tanzania. The Center is due to open in 3 years. The Center will use the donated funds to cover their annual equipment and testing costs. They will invest the funds at 4.25%, effective, into a perpetuity. They will withdraw the first perpetuity payment in exactly 3 years when the center opens. Calculate the size of the annual withdrawals.

Let us first draw out a timeline for this problem:



Next, determine which part to start with. In this case, because we know the initial deposit (PV_1), start with part 1 (the deferral period).

Part 1 (the deferral period):

B/E	P/Y	C/Y	N_1	I/Y	PV_1	PMT_1	FV_1
--	1	1	3×1=3	4.25	+1,000,000	0	**CPT** −1,132,995.52

It does not matter what we enter in B/E above because there are no payments. Let us now enter the ending value for the deferral period (FV_1) into the starting value for the perpetuity (PV_2).

Part 2 (the perpetuity):

B/E	P/Y	C/Y	N_2	I/Y	PV_2	PMT_2	FV_1
BGN	1	1	1000×1=1000	4.25	+1,132,995.52	**CPT** −46,189.27	0

Notice that BGN is on in part 2. This is because we started part 2 (the perpetuity) right when the first payment was withdrawn.

Conclusion: The Moshi Medical Center will receive $46,189.27 annually to cover equipment and testing costs starting in 3 years when the center opens.

KEY TAKEAWAYS FOR PERPETUITIES

Key Takeaways for Perpetuities

When entering perpetuities into the BAII Plus:

- FV is always equal to 0 for perpetuities.
- We always use 1,000 years for perpetuities.
- This gives N = 1000 × P/Y
- It does not matter what sign we use for PV or PMT (as we are computing the other one)[9]

When using formulas:

- For ordinary perpetuities, $\text{PMT} = \text{PV} \times i$
- For perpetuities due, $PV_{due} = \dfrac{PMT}{i} + PMT$

For deferred perpetuities:

- Run the deferral period until the first payment occurs
- Turn BGN on for part 2
- Set $PV_2 = FV_1$
- Set $FV_2 = 0$.

YOUR OWN NOTES

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THE FOOTNOTES

Notes

1. Information thanks to https://languages.oup.com/google-dictionary-en/ ; https://www.investopedia.com/terms/p/perpetuity.asp ; https://en.wikipedia.org/wiki/War_bond; https://www.advisor.ca/tax/estate-planning/design-a-dynasty-with-perpetual-trusts/; https://www.investopedia.com/terms/t/trust-fund.asp; https://thephilanthropist.ca/original-pdfs/Philanthropist-21-3-370.pdf;
2. Historically, governments issued bonds to raise money to meet the growing costs of war. Some of these bonds lasted in perpetuity.
3. Suppose a company invests a certain initial amount of money with the hope/expectation that they will earn a return on their investment in the form of regular income from the business (possibly for an indefinite amount of time). If we assume the returns last for an indefinite amount of time, we treat this problem like a perpetuity as well.
4. A perpetuity, technically, lasts forever. In the calculator, the closest to 'forever' we should enter for a number of years is 1,000. This number of years will work with all types of calculations in the TVM keys.
5. The same principle applies as in ordinary perpetuities, the balance in the scholarship fund never increases nor decreases in value. This is because after a payment has been withdrawn from the account, interest is earned on the remaining balance in the scholarship fund. The payments are calculated such that the after the interest has been earned, the balance in the account increases back to its original value (PV).
6. We, again, need to be careful if the number of payments per year (P/Y) is not equal to the number of compounding periods per year (C/Y). If that is the case, we need to calculate the equivalent interest rate with the number of compounding periods equal to the number of payment intervals for the perpetuity.
7. They deposit the money once it's received but do not need to start withdrawing from the account until the non-profit opens. If we assume that the non-profit will remain open indefinitely, then we assume that they will need withdrawals to cover their costs for an undetermined amount of time as well. We treat this problem as a perpetuity.
8. In that case, the repayments of the initial investment are delayed (deferred) for a certain number of years. If we assume that there is no fixed end date to the

business, we treat this problem as a perpetuity where the returns continue on indefinitely.

9. If ever both were being entered into the BAII plus - we would need to use opposite signs for PV and PMT

5.9 Preferred Shares

Learning Outcomes

Calculate the fair market value and gain (or loss) when purchasing preferred shares.

When a company is looking to raise funds, they can issue **preferred shares**. A preferred share has no **maturity date** (no expiration date or fixed term) but will stop regular payments if the company stops making a profit or goes out of business. Because we usually have no way of knowing when a business will end or stop making profits, we will treat preferred shares like perpetuities and assume the profits (and therefore regular payments) will continue indefinitely (forever).

Let us first understand some of the terminology:

- **Dividend**: The regular payment (PMT) paid by the issuer of the preferred share. The value of this dividend remains fixed and does not change through the lifetime of the share.
- **Price**: The selling price of the share (PV). This is the fair market value (ie: what you are willing to pay for the share).
- **Market Rate**: The current interest rate (I/Y).

See the sections below for the key formulas, tips and examples related to calculating values for preferred shares.

CALCULATING FAIR MARKET VALUES USING THE FORMULAS

To calculate any values for preferred shares, you can use the same formulas as for perpetuities. If the dividend is awarded at the end of the each payment interval, and if you would like to know the fair market value of the share, use the following formula:

$$PV = \frac{PV}{i}$$

If you would like to know the fair market value of the preferred share and the dividends are awarded at the start of each payment interval, use the following formula:

$$PV_{due} = \frac{PMT}{i} + PMT$$

Let us now look at Kiana's purchase of preferred shares and determine how much she should be willing to pay to purchase the preferred shares.

Example 5.9.1

Kiana purchases 100 Koodo preferred shares. Koodo pays a dividend of $0.80 every quarter on their preferred shares. The next dividend will be paid out in 3 months. If Kiana wants to earn a minimum of 5% compounded quarterly on her investment, how much should she be willing to pay for the 100 shares?

Let us first calculate the periodic rate (*i*) using the nominal rate (j_4 = 2.75%):

$$i = \frac{5}{4} = \frac{0.05}{4} = 0.0125$$

Because the first payment will be paid out at the end of the first interval (in 3 months), we use the following formula to calculate PV (the price per share):

$$PV = \frac{PMT}{i}$$

Inputting *i* = 0.0125 and PMT = $0.80 into the formula gives the following price per share:

$$PV = \frac{\$0.80}{0.0125} = \$64$$

Kiana should be willing to pay $64 per share. To calculate the price she should pay for all 100 shares, multiply by the 100 shares:

$$\text{Price for 100 Shares} = \$64 \times 100 = \$6,400$$

Conclusion: Kiana should pay $6,400 for the 100 Koodo preferred shares.

Example 5.9.2

Let us again look at Kiana's purchase of 100 Koodo preferred shares (from Example 1). What if the first dividend were paid immediately instead of in 3 months? How much should Kiana be willing to pay for the 100 shares?

Because the first payment will be paid out immediately, we use the following formula to calculate PV (the price per share):

$$PV = \frac{PMT}{i} + PMT$$

Inputting i = 0.0125 and PMT = \$0.80 into the formula gives:

$$PV = \frac{\$0.80}{0.0125} + \$0.80 = \$64.80$$

Kiana should be willing to pay \$64.80 per share. To calculate the price she should pay for all 100 shares, multiply by the 100 shares:

$$\text{Price for 100 Shares} = \$64.80 \times 100 = \$6,480$$

Conclusion: Kiana should pay \$6,480 for the 100 Koodo preferred shares.

CALCULATING FAIR MARKET VALUES USING THE BAII PLUS

We can also use your BAII Plus calculator to calculate the fair market value of preferred shares. We use the same 'tricks' as for perpetuities:

- N = 1000×P/Y. Use 1000 years when calculating the number of payments (N)[1].
- I/Y = Nominal Interest Rate (the current interest rate).
- FV = 0. You can never take the principal out. You only ever get paid the regular dividends.
- PMT = Regular Dividend Amount. We will enter this amount as positive because it is the amount the share holder receives[2].
- **CPT** PV. Calculate the fair market value of the share (PV).

Example 5.9.3

Calculate Kiana's price per share for shares from Example 1 using the BAII Plus:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	4	4	1000×4=4000	5	**CPT** −64	+0.80	0

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text. You can view it online here:
https://pressbooks.bccampus.ca/businessmathematics/?p=3360#h5p-74

Conclusion: Kiana should pay $64 per share the Koodo preferred shares (the same prices as in Example 1).

Example 5.9.4

Calculate Kiana's price per share for shares from Example 2 using the BAII Plus:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
BGN	4	4	1000×4=4000	5	**CPT** −64.80	+0.80	0

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https://pressbooks.bccampus.ca/businessmathematics/?p=3360#h5p-75

Conclusion: Kiana should pay $64.80 per share the Koodo preferred shares (the same prices as in Example 2).

CALCULATING THE GAIN OR LOSS FROM PREFERRED SHARES

We calculate the gain or loss from the sale of preferred ways in the same way as for bonds:

$$\text{Gain (or Loss)} = \text{Selling Price} - \text{Purchase Price}$$

When the selling price is higher than the purchase price, we gain money on the sale of the shares. When the selling price is lower than the purchase price, we loose money on the sale.

Example 5.9.5

Kiana paid $64.00 per share when she bought 100 Koodo shares. Kiana would like the sell the shares several years later when interest rates have risen to 6.4%. Assume the next dividend of \$0.80 will be paid in 3 months. Calculate Kiana's gain (or loss) on the sale of the 100 shares.

Check Your Knowledge for Example 5.9.5

We first need to calculate the fair market value of the shares. Which value do we calculate do this?

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Which formula could we use to calculate the fair market value for this question?

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Determine what to enter into the BAII Plus to calculate this value:

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An interactive H5P element has been excluded from this version of the text. You can view it online here: https://pressbooks.bccampus.ca/businessmathematics/?p=3360#h5p-81

Kiana can sell each share for $50 when she goes to sell them several years later. We can use this price to calculate the gain or loss on Kiana's 100 shares:

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Conclusion: Kiana will lose $1,400 on the sale of her 100 Koodo shares.

KEY TAKEAWAYS FOR PREFERRED SHARES

Key Takeways for Preferred Shares

Calculating the price per share in the BAII Plus:

- FV = 0. FV is always equal to 0.
- N = 1000 × P/Y. Always use N = 1,000 years × P/Y.
- PMT = Dividend value. Enter PMT as positive[3].
- CPT PV. Fair Market Value = PV. [4]

Price per Share formulas:

- Payments at end of interval: $PV = \frac{PV}{i}$
- Payments at beginning of interval: $PV_{due} = \frac{PMT}{i} + PMT$

Gain or loss per share:

$$\text{Gain (or Loss)} = \text{Selling Price} - \text{Purchase Price}$$

Total gain or loss:

$$\text{Total Gain (or Loss)} = \text{Gain (or Loss) per Share} \times \text{Number of Shares}$$

YOUR OWN NOTES

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
- These notes are for you only (they will not be stored anywhere)
- Make sure to download them at the end to use as a reference

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THE FOOTNOTES

Notes

1. Again, this is because the payments continue on forever and 1000 years is the closest to forever that we can input in the BAII Plus
2. It does not actually matter what sign we input for PMT because we will be calculating PV each time. If ever we were inputting both PV and PMT into the BAII Plus, we would need to make sure they are opposite in sign.
3. It does not matter what sign we use for PMT (as we are computing the PV).
4. If ever both were being entered into the BAII plus - we should enter PMT as positive and PV as negative or at least use opposite signs for PV and PMT.

5.10 Mortgages

Learning Outcomes

Calculate the payment size, balance, principal repaid or interest charged during a mortgage and understand how to renew a mortgage.

It is often necessary to take out a mortgage when purchasing a home or property. A mortgage is usually a long-term (roughly 25 year), **general** (P/Y ≠ C/Y), **ordinary** annuity (PMTs at END of interval). Buyers can choose a **fixed** or **variable rate** mortgage. A fixed rate mortgage means that the interest rate charged remains fixed for the **mortgage term**. A variable rate mortgage means that the interest rate varies throughout the mortgage term. In general, the buyer (mortgage holder) is required to pay down some of the balance owing (**principal**) on the mortgage in addition to the interest charges each interval.

HISTORY OF HOW MORTGAGES EVOLVED OVER THE YEARS

In the 1900's, home buyers who took out a mortgage only paid the interest owed each month. The buyer would save up while making these interest-only payments and fully repay the mortgage when they had enough saved.[1]

Major world events (like the first world war & great depression) changed this practice. Large numbers of people were unable to pay their mortgages. Because of this, buyers were then required to pay some of the balance owing (**principal**) each payment interval in addition to the interest owed each month. To further protect lenders, in 1946, the "Canada Mortgage and Housing Corporation(CMHC)" was created. The CMHC insures buyers' mortgages if the buyer puts down less than a minimum % down payment.[2] This protects lenders if a buyer **defaults** on their mortgage (can't pay their mortgage).

Before the 1970/80's, buyers would be guaranteed a **fixed interest rate** for the entire duration of their mortgage. This changed after the interest rate inflation in the 1970's and 80's (interest rates hit a record high of 21.46%). Banks who had previously locked in interest rates with buyers were missing out on thousands of potential dollars in interest when buyer were locked in at rates as low as 6.9% and when the current interest rate was 21.46%. After the interest rate inflation of the 1970's and 1980's, banks created "**mortgage terms**."

MORTGAGE TERMS

A mortgage term is the length of time your mortgage agreement and interest rate will be in effect.[3] Mortgage terms are, most often, 5 years in length but can vary anywhere from 6 months to 10 years in length. If the buyer chooses a fixed-rate mortgage, they are guaranteed a fixed interest rate for the duration of the mortgage term. After the term is 'up' (after the time period defined by the term has passed), the buyer negotiates a new interest rate with the lender (bank). The buyer **renews** their mortgage.

Basically, a new mortgage is drawn up at the end of each term when the buyer renews their mortgage. The buyer can pay off part (or all) of the balance owing with a lump-sum payment. The buyer can move their mortgage to another bank. The buyer is charged interest at the new interest rate and pays this interest on the current balance (principal) owing on the mortgage.

If each term is 5 years in length, a buyer will renew their mortgage roughly five times on a 25-year mortgage. The full length of the mortgage (ex: 25 years) is the **amortization period.**

CONSTRUCTING AMORTIZATION TABLES

Amortization is "the process of paying off debt over time with regular installments of interest and principal sufficient to repay the loan in full by its **maturity date.**[4]"

We can create an **amortization table** (or schedule) to show the amount of principal and interest that make up each payment. The table also shows the balance owing after each payment. The table can run until the loan (or mortgage) if fully paid off or to the end of the term.

Let us now look at an example of an amortization schedule for Kerry — she just bought a car and borrowed some money from her line of credit to do so.

Example 5.10.1

Kerry purchases a new Toyota Rav4. She borrows $16,000 from her line of credit for the purchase. She is charged 3% compounded monthly on the line of credit. Kerry wants to repay her line of credit with 6 monthly payments. The first payment will be in one month. Construct an amortization schedule and use it to answer how much interest she will pay in total and what the size of her final payment will be. Assume Kerry owed nothing on her line of credit before she borrowed the $16,000.

In order to construct the amortization schedule, we need to determine the size of Kerry's monthly payments:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	12	6	3	+16,000	**CPT** −2,690.05	0

Let's round Kerry's payment up to the next dollar[5] which means that Kerry will pay $2,691 per month.

Because we round up Kerry's payments, this means that she will overpay by roughly $0.95 per month. This makes her final payment slightly smaller. Let us now construct the amortization schedule to determine the size of this final payment as well as the interest paid and balance outstanding each month.

Line of Credit Amortization Table

Payment #	Payment (PMT)	Interest Paid (INT)	Principal Repaid (PRN)	Ending Balance (BAL)
1	$2,691	$16,000×0.0025 =$40	$2,691−$40 =$2,651	$16,000−$2,651 = $13,349
2	$2,691	$13,349×0.0025 =$33.37	$2,691−$33.37 =$2,657.63	$13,349−$2,657.63 = $10,691.37
3	$2,691	$10,691.37×0.0025 =$26.73	$2,691−$26.73 =$2,664.27	$10,691.37−$2,664.27 =$8,027.10
4	$2,691	$8,027.10×0.0025 =$20.07	$2,691−$20.07 =$2,670.93	$8,027.10−$2,670.93 =$5,356.17
5	$2,691	$5,356.17×0.0025 = $13.39	$2,691−$13.39 =$2,677.61	$5,356.17−$2,677.61 =$2,678.56
6	$2,691[6]	$2,678.56×0.0025 =$6.70	$2,691−$6.70 =$2,684.30	$2,678.56−$2,684.30 =−$5.74

Key Takeaways for the Above Amortization Table

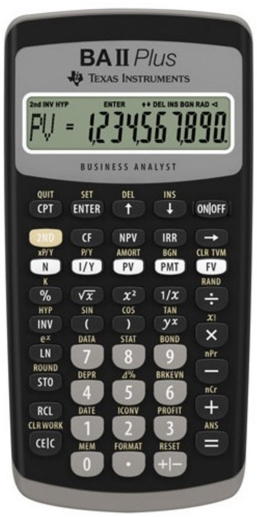
- Payment Amount = PMT = $2,691 for this example
- Interest Paid (INT) = Previous Ending Balance × i
- Where i = periodic rate = $^{j_m}/_{\mathrm{m}} = {}^{0.03}/_{12} = 0.0025$
- Principal Repaid (PRN) = Payment Amount − Interest Paid
- Ending Balance (BAL) = Previous Ending Balance − Principal Repaid
- Final Payment = Payment (PMT) + Final Ending Balance = $2,691+(−$5.74) = $2,685.26
- Add up all of the interest paid to calculate the total interest paid:
  Total Interest = $40 + $33.37 + $26.73 + $20.07 + $13.39 + $6.70 = $140.26
- You could also construct this table in Excel (click here to download the Excel file).

Conclusion: Kerry will pay $140.26 in interest total and make a final payment of $2,685.26 to pay off her line of credit.

AMORTIZATION USING AMRT IN THE BAII PLUS

It can take a long time to construct an amortization table, especially for long-term loans. You can instead construct the table in Excel or use the BAII Plus' **AMRT** (amortization) menu. In order to access the AMRT menu in your BAII Plus, you need to hit **2ND** **PMT** after you have already entered all values in the **TVM keys** and have rounded up your payment (PMT) and re-entered it into your calculator as a negative value. These steps are written out below:

Steps to Using the AMRT Menu in Your BAII Plus



1. Compute the missing value in the TVM keys (ex: PMT)
2. Input the rounded up payment value. Then make it negative and re-enter it using: +|− PMT
3. Hit 2ND PMT to enter the AMRT menu
4. Enter in a value for P1 and hit ENTER ↓
5. Enter in a value for P2 and hit ENTER ↓
6. Scroll through the AMRT menu using ↑ ↓

   In the above steps, we did not explain what P1 and P2 mean. Let's now make sense of these values as well as the rest of the values given by the AMRT menu.

Understanding the Values in the AMRT Menu:

- P1: Starting payment in period in question
- P2: Ending payment in period in question
- BAL: Outstanding Balance at end of period in question
- PRN: Principal Repaid during period in question
- INT: Interest Paid during period in question

We have not yet clearly defined what the "period in question" means. This is because this period can vary. For monthly payments, the period in question is the starting and ending month numbers. Some examples are given below for P1 and P2 values.

Examples of "Periods in Question" for Monthly Payments

1. The first year of a mortgage ⇒ $P_1=1$, $P_2=12$ (there are 12 months in the first year)
2. The first month of a mortgage ⇒ $P_1=1$, $P_2=1$ (period is just month 1)

3. The third year of a mortgage ⇒ P_1=25, P_2=36 (months 25 to 36 are in the third year)
4. The third month of a mortgage ⇒ P_1=3, P_2=3 (just period 3)
5. The first three years of a mortgage ⇒ P_1=1, P_2=36 (payments 1 to 3×12=36)

Example 5.10.2

Maksim purchases an apartment in New Westminster. He pays $600,000 less a 20% down payment. He takes out a 25-year mortgage with a 5-year term. He is charged 2.95%, compounded semi-annually. He makes monthly payments with the first payment in one month. How much interest will he pay during the first 5-year term? Assume that his payments are rounded up to the next dollar.

Step 0: Determine the amount Maksim borrows (PV):

$$\begin{aligned}\text{Amount Borrowed} &= \text{Price} - \text{Down Payment}\\ &= \$600,000 - \$600,000 \times 0.20\\ &= \$600,000 \times (1 - 0.20)\\ &= \$480,000\end{aligned}$$

Step 1: Determine the size of Maksim's monthly mortgage payments:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	2	25×12=300	2.95	+480,000	**CPT** −2,259.28	0

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Step 2: Round up the payment and re-enter as a negative value: 2260 **+|−** **PMT**

Step 3: Access the AMRT menu: **2ND** **PMT**

Step 4: Input P_1: 1 **ENTER** **↓**

Step 5: Input P_2: 60 **ENTER** **↓**

Step 6: Scroll down to INT: **↓** **↓** **↓**

Conclusion: Maksim will pay $65,436.89 in interest during the first 5 years of his mortgage.

Wondering why we used P_1=1 and P_2=60? There are 60 months in the first 5 years, starting with month one and ending with month 60. Also see the table in the section below below.

P1 & P2 Values for First 5 years of Monthly Payments

We want to calculate the interest for the first 5 years of the mortgage (the first term). The first month in this time-period is month 1 (the start of the mortgage). The last month will be month 60 (=5×12). See the table below for the month numbers for the first 5 years.

Year	P1	P2	Month Numbers (P1 to P2)
1	1	12	The first year contains months 1 to 12
2	13	24	The 2nd year contains months 13 to 24
3	25	36	The 3rd year contains months 25 to 36
4	37	48	The 4th year contains months 37 to 48
5	49	60	The 5th year contains months 49 to 60

Example 5.10.3 — How much of the first mortgage payment will be interest?

Step 0—2: Make sure the values from Example 2 entered into the TVM keys (N, I/Y, PV, PMT, FV) in your BAII Plus.

Step 3: Access the AMRT menu: 2ND PMT

Step 4: Input P_1 (the first payment 'starts' in month 1): 1 ENTER ↓

Step 5: Input P_2 (the first payment 'ends' in month 1): 1 ENTER ↓

Step 6: Scroll down to INT: ↓ ↓ ↓

Conclusion: Maksim will pay $1,172.82 in interest during the first month of his mortgage.

Example 5.10.4— How much will the balance owing be reduced by with the first payment?

In this case, we want to calculate the principal repaid (this is the amount we reduce the balance owing by with each payment that we make). Make sure all values from Example 2b are still in your calculator (Steps 0 to 5). Just scroll back up ↑ to PRN.

Conclusion: Maksim will reduce his balance owing by $1,087.19 with his first mortgage payment.

Example 5.10.5— How much interest does Maksim repay in the first year?

There are 12 months of payments in the first year, starting at payment 1:

P_1 = 1
P_2 = 12
INT = $13,896.99

Conclusion: Maksim will pay $13,896.99 in interest in the first year.

Example 5.10.6 — How much principal does Maksim repay in the fifth year?

The fifth year starts at month 49 and ends at month 60:

P_1 = 49
P_2 = 60
PRN = $14,866.29

Conclusion: Maksim will reduce his balance owing by $14,866.29 in the fifth year.

Example 5.10.7 — How much Maksim still owe at the end of five years?

If we want the balance owing, it only matters what is entered into P2. This is because BAL (balance) is the amount owing at the end of payment P2. For this reason, we can just leave the values from Example 2e in the calculator and scroll to BAL:

P_1 = 49
P_2 = 60
BAL = $409,836.89

Conclusion: Maksim will owe $409,836.89 at the end of five years.

RENEWING MORTGAGES

At the end of a mortgage term, the mortgage holder renews their mortgage (or refinances the mortgage if they borrow more money).

When the mortgage holder renews their mortgage, their terms and interest rate will most likely be changed. Use this new rate and use the number of years remaining to determine the size of the new mortgage payments.

Example 5.10.8

Let us assume Maksim has made 5 years of mortgage payments and it is now time for Maksim's to renew his mortgage. Maksim renews his mortgage for another 5 years at 3.5% compounded semi-annually. What is the size of Maksim's new mortgage payments? Round up to the next dollar.

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	2	20×12=240	3.5	+409,836.89	**CPT** −2,371.57	0

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Conclusion: Maksim will pay $2,372 per month (round the payments up).

CALCULATING THE FINAL PAYMENT

The final payment using the BAII Plus AMRT method can be calculated the same way as when we using the amortization table (see Example 1). The only differences are that we enter the final payment number into P2 and scroll down to BAL to determine the final balance owing. We then use the same formula as before:

$$\text{Final Payment} = \text{Regular Payment Size (PMT)} + \text{Final Ending Balance}$$

Note: the ending balance will be negative. That means that when we add that negative number to the regular payment size, the payment size drops in value.

Example 5.10.9

Let us continue on with Example 5.10.9. Let us assume Maksim continues to pay 3.5% compounded semi-annually for the entire 20 years remaining in his mortgage. What will be the size of Maksim's final payment if this were true?

Step 1: Make sure the values from Example 3 entered into the TVM keys.

Step 2: Round up the payment and re-enter as a negative value: 2372 +|− PMT

Step 3: Access the AMRT menu: 2ND PMT

Step 4: Input P_1 (input any value up to 240): 1 ENTER ↓

Step 5: Input P_2 (final payment is the 240$^{\text{th}}$ payment): 240 ENTER ↓

Step 6: Scroll down to BAL: ↓

Step 7: Calculate the final payment size using BAL=−147.20:

$$\text{Final Payment} = \$2372 + (-\$147.20) = \$2,224.80$$

Conclusion: Maksim will make a final payment of $2,224.80 at the end of 20 years.

CALCULATING THE INTEREST CHARGED USING THE FORMULA

There are two ways to calculate the total interest charged on a mortgage. We can calculate the difference between the money out and money in or we can use the AMRT menu and read off the INT values. Let us first step through taking the difference between money out and in calculation:

$$\text{Interest Charged} = \text{Money Out} - \text{Money In} = \$\text{OUT}-\$\text{IN}$$

To determine $ OUT, where be sure to add up ALL payments and be careful of the final payment — it is often smaller than the rest:

$$\$\ \text{OUT} = \text{Sum of All Mortgage Payments}$$

To determine $ IN, total all money borrowed. If the mortgage holder borrows more money at some point during the mortgage (if they refinance), then also include that amount in the $ IN calculation:

$$\text{Money In } (\$\ \text{IN}) = \text{Total Amount Borrowed}$$

Let us now determine how much interest Maksim will be charged on this mortgage.

Example 5.10.10

Let us continue on with this example. Let us assume Maksim continues to pay 3.5% compounded semi-annually for the entire 20 years remaining in his mortgage. If this is true, how much interest does Maksim pay in total on his mortgage?

Let us first calculate the money out ($ OUT):

$$\begin{aligned}\text{Money Out (\$ OUT)} &= \text{Sum of All Mortgage Payments}\\ &= \$2,260 \times 60 + \$2372 \times 239 + \$2,224.80\\ &= \$135,600 + \$566,908 + \$2,224.80 = \$704,732.80\end{aligned}$$

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://pressbooks.bccampus.ca/businessmathematics/?p=887#h5p-90

Next, let's determine the money in ($ IN):

$$\begin{aligned}\text{Money In (\$ IN)} &= \text{Total Amount Borrowed}\\ &= \$480,000\end{aligned}$$

Taking the difference between the money out and in gives:

$$\text{Interest Charged} = \$704,732.80 – \$480,000 = \$224,732.80$$

Conclusion: Maksim will be charged $224,732.80 in interest over the 25 years.

CALCULATING THE INTEREST CHARGED USING THE AMRT MENU

Let us now step through how to use the AMRT menu results to calculate the total interest charged on a mortgage.

For each term in the mortgage (or for each period where the interest rate is fixed), calculate the total interest paid during that term by doing the following:

P_1 = 1
P_2 = Final Payment # in term
INT = Total interest paid during that term

Redo this calculation for each term in the mortgage and add up all of these values to determine the total interest paid over the entire mortgage. Let us revisit Maksim's mortgage a final time in this section to understand how to perform this calculation.

Example 5.10.11 — Calculate the total interest charged using the AMRT menu

We have already calculated the total interest charged for the first term in Maksim's mortgage. We found that Maksim paid $65,436.89 in interest during the first 5 years of his mortgage.

To determine the amount of interest Maksim will pay in the remaining 20 years[7], let us look back at Example 4 and do the following:

1. Make sure the values from Example 4 are still in the TVM keys (they should be)
2. Access the AMRT menu: 2ND PMT
3. Make sure P_1=1 (it should still be 1)
4. Make sure P_2=240 (it should still be 240)
5. Scroll down to INT: ↓ ↓ ↓ = $159,295.91

Use this interest amount as well as the $65,436.89 to determine the total interest charged:

Total Interest Charged = $65,436.89 + $159,295.91 = $224,732.80

Conclusion: Maksim will be charged $224,732.80 in interest over the 25 years (the same amount as calculated in Example 5).

YOUR OWN NOTES

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
- These notes are for you only (they will not be stored anywhere)
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THE FOOTNOTES

Notes

1. Information thanks to the Financial Services Commission of Ontario: https://www.fsco.gov.on.ca/en/mortgage/Pages/history.aspx
2. Currently, the CMHC insures mortgages with “mortgage default insurance” if the less than a 20% down payment is made at the start of the mortgage.
3. Information thanks to https://www.canada.ca/en/financial-consumer-agency/services/financial-toolkit/mortgages/mortgages-2/6.html
4. https://www.investopedia.com/terms/a/amortization.asp
5. For mortgages and loans, banks round mortgage payments up to the next dollar or the next cent. The final mortgage (or loan) payment is smaller as a result of the regular overpayments.
6. This will not be the actual size of the final payment. The final payment will actually be equal to this value minus the overpayment (final ending balance) =$2,691−$5.74 = $2,685.25
7. Normally, Maksim would renew his mortgage every 5 years. In this case, the interest rate would change every five years and there would roughly 5 different interest rate calculations (one per term). For Maksim's mortgage example, we assumed Maksim would be charged 3.5% for the remaining 20 years of his mortgage. Although this is unlikely, it simplified our calculations (this section is already incredibly long and the calculations would be similar every time Maksim renews).

5.11 The 32% Rule and Bi-Weekly Payments

Learning Outcomes

Use the 32% rule to determine the maximum mortgage amount. Also, calculate the size of accelerated biweekly mortgage payments or the savings in interest.

There are two more topics to examine to conclude our section on Mortgages. The first is the 32% rule, which states that no more than 32% of your gross monthly can go towards housing costs. The second (and final topic) will be how to calculate accelerated bi-weekly payments and savings when choosing to make accelerated bi-weekly payments.

THE 32% RULE

Your housing costs shouldn't be more than 32% of your gross monthly income.[1] Housing costs include your mortgage payment, property taxes, heating costs and half of your strata (condo) fees.[2]

$$\text{PMT} + \text{Prop Taxes} + \text{Heat Costs} + 0.5 \times \text{Strata Fees} \leq 0.32(\text{Gross Monthly Income})$$

We can rework this formula to determine the maximum allowable monthly mortgage payment:

$$\begin{aligned} \text{PMT} &\leq 0.32(\text{Gross Monthly Income}) - (\text{Prop Taxes} + \text{Heat Costs} + 0.5 \times \text{Strata Fees}) \\ &\leq 0.32(\text{Gross Monthly Income}) \text{ - Prop Taxes - Heat Costs} - 0.5 \times \text{Strata Fees} \end{aligned}$$

Let us now use this rule to determine how much Brenda and Huong will be allowed to borrow when they go to purchase an apartment in Central Surrey.

Example 5.11.1

Huong and Brenda are looking to purchase a 2-bedroom apartment in Central Surrey. The sisters' combined gross income is $150,000. The property taxes on the 2 bedroom apartment are $3,600/year. The average heating cost is $43/month. The strata fee for the apartment is $500/month. What is the largest mortgage payment Brenda and Huong would be allowed to make per month?

Let us use the 32% rule to determine Huong and Brenda's maximum allowable mortgage payment. To use the rule, we need all housing expenses monthly. Let us calculate the equivalent monthly amounts for Huong and Brenda's income and their potential property taxes:

$$\text{Gross Monthly Income} = \frac{\$150,000}{12} = \$12,500$$

$$\text{Monthly Property Taxes} = \frac{\$3,600}{12} = \$300$$

This gives the following for the maximum allowable mortgage payment:

$$\begin{aligned} PMT &\leq 0.32(\text{Gross Monthly Income}) - \text{Prop Taxes} - \text{Heat Costs} - 0.5(\text{Strata Fees}) \\ &\leq 0.32(\$12,500) - \$300 - \$43 - 0.5(\$500) \\ &\leq \$3407 \end{aligned}$$

Conclusion: Huong and Brenda can pay no more than $3,407 per month on their mortgage payment.

Example 5.11.2

Huong and Brenda have $100,000 saved up for a down payment on an apartment. The current interest rate is 2.4% compounded semi-annually for a fixed rate 25 year mortgage with a 5-year term. What is the most expensive apartment Brenda and Huong would be allowed to purchase?

Let us use the maximum allowable monthly mortgage payment (PMT) from Example 1 to calculate the maximum Huong and Brenda are allowed to borrow (PV):

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	2	25×12=300	2.4	**CPT** +769,071.85	−3,407	0

Huong and Brenda can borrow $769,071.85. To determine price of the most expensive apartment, include the down payment:

$$\begin{aligned} \text{Max Allowable Price} &= \text{Max Amount Borrowed} + \text{Down Payment} \\ &= \$769,071.85 + \$100,000 \\ &= \$869,071.85 \end{aligned}$$

Conclusion: Huong and Brenda can buy an apartment costing up to $869,071.85.

BI-WEEKLY & ACCELERATED BI-WEEKLY PAYMENTS

A bi-weekly payment is a payment that occurs once every 2 weeks. Bi-weekly payments are popular because people are commonly paid bi-weekly. There are 26 bi-weekly payment periods in a year.

When paying bi-weekly, people can choose the **accelerated bi-weekly** option. When they choose an accelerated bi-weekly mortgage, they pay half of the normal monthly payment every two weeks. When making this choice, they end up making 2 extra bi-weekly payments per year. This happens because for two of the twelve months in the year, the borrower will receive three paychecks during that month. This happens in months where their payday lands on the following days[3]:

- The 1st, 15th and 29th of the month
- The 2nd, 16th and 30th of the month

Those two extra bi-weekly payments equal to one extra full-sized monthly mortgage payment. This overpayment by one full-sized mortgage payment per year will lead to savings in interest paid on the mortgage and the time required to pay off the mortgage.

Let us look again to Huong and Brenda and their mortgage options when they purchase an apartment in Central Surrey.

Example 5.11.3

Huong and Brenda have found the perfect apartment! It costs $850,000 and they will make a $100,000 down payment. Now they just need to decide whether to make monthly mortgage payments or accelerated bi-weekly mortgage payments.

Help them decide by determining the size of the mortgage payments for both options as well as the potential time and money saved if Huong and Brenda make accelerated bi-weekly payments (if they can afford it)!

Let us start by calculating the size of Huong and Brenda's monthly mortgage payments.

Example 5.11.3a – The Size of Huong and Brenda's Monthly Payments

Step 0: Determine the amount Huong and Brenda will borrow:

$$\begin{aligned}\text{Amount Borrowed} &= \text{Selling Price} - \text{Down Payment}\\ &= \$850,000 - \$100,000\\ &= \$750,000\end{aligned}$$

Step 1: Determine the size of the regular monthly mortgage payments (assume Huong and Brenda will still have a 25-year mortgage and be charged 2.4%, compounded semi-annually):

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	2	25×12=300	2.4	+750,000	CPT −3,322.51	0

Conclusion: If Huong and Brenda make monthly payments, they will pay $3,323 per month.

Example 5.11.3b – Total payments Per Year (Monthly)

Let us now calculate how much Huong and Brenda will make in mortgage payments per year:

$$\text{Total Payments per Year} = \$3323 \times 12 = \$39,876$$

Conclusion: Huong and Brenda will make $39,876 in mortgage payments each year if they pay monthly.[4].

Example 5.11.3c – Total Interest Paid with Monthly Payments

Now that we have the size of Huong and Brenda's monthly payments, we can calculate the interest they will pay over the 25 years. We will assume, to avoid making this example too long, that the interest rate will remain fixed at 2.4% for the entire 25 years.

Step 1: Make sure all values from example 3a are still in your BAII Plus (TVM keys).

Step 2: Round up the payment and re-enter as a negative value: 3323 [+|−] [PMT]

Step 3: Access the AMRT menu: [2ND] [PMT]

Step 4: Input P_1: 1 [ENTER] [↓]

Step 5: Input P_2: 300 [ENTER] [↓]

Step 6: Scroll down [↓] [↓] [↓] to INT: −246,699.76

Conclusion: Huong and Brenda will pay $246,699.76 in interest if they choose to make monthly payments.

Example 5.11.3d – Time-Savings with Accelerated Bi-Weekly

When making accelerated bi-weekly payments, we divide the regular monthly payments in half:

$$\text{Accelerate Bi-Weekly Payment} = \frac{\$3,323}{2} = \$1,661.50$$

We then enter this payment amount into the BAII Plus:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	2	**CPT** 583.157	2.4	+750,000	−1,661.50	0

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https://pressbooks.bccampus.ca/businessmathematics/?p=926#h5p-91

It will take them 583 full-sized and one smaller payment to pay off the mortgage with accelerated bi-weekly payments. This will still be 584 payments (even though the last one is smaller, it still counts as a payment). Let us calculate how many years that is:

$$\text{Number of Years} = \frac{584}{26} = 22.46 \text{ years}$$

It will take them 22.46 years to pay of their mortgage with accelerated bi-weekly payments. Let's compare that to the time required to pay off the mortgage with monthly payments:

$$\text{Time Saved} = 25 - 22.46 = 2.54 \text{ years}$$

Conclusion: Huong and Brenda will save 2.54 years if they choose to make bi-weekly payments.

Example 5.11.3e – Additional Amount Paid per Year with Accelerated Bi-Weekly

Let us now calculate how much Huong and Brenda will make in bi-weekly payments per year:

$$\text{Total Payments per Year} = \$1661.50 \times 26 = \$43,199$$

Compare this amount to the total paid with monthly payments to determine how much more they will pay per year if they make accelerated bi-weekly payments:

$$\text{Extra Amount Paid} = \$43,199 - \$39,876 = \$3,233$$

Conclusion: Huong and Brenda pay an extra $3,233 in mortgage payments if they make accelerated bi-weekly payments. Notice that this is exactly equal to one monthly payment.

Example 5.11.3f — Savings in Interest with Accelerated Bi-Weekly

Steps 0-2: Check that all values for Example 3d are still in your BAII Plus.

Step 3: Access the AMRT menu: 2ND PMT

Step 4: Input P_1: 1 ENTER ↓

Step 5: Input P_2: 584 ENTER ↓ (There are 584 bi-weekly payments in total).

Step 6: Scroll down ↓ ↓ ↓ INT: −218,915.65

Huong and Brenda will pay $218,915.65 in interest if they make accelerated bi-weekly payments. We can use this amount to figure out their savings in interest with this payment plan:

$$\text{Money Saved} = \$246,699.76 - \$218,915.65 = \$27,784.11$$

Conclusion: Huong and Brenda will save $27,784.11 in interest if they pay with accelerated bi-weekly payments instead of monthly payments.

Example 5.11.3g — What Should Huong and Brenda Decide?

If Huong and Brenda can afford to pay the extra $3,233 per year, then they should choose the accelerated bi-weekly payment option and save 2.54 years and $27,784.11 in interest when repaying their mortgage!

YOUR OWN NOTES

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?

- These notes are for you only (they will not be stored anywhere)
- Make sure to download them at the end to use as a reference

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https://pressbooks.bccampus.ca/businessmathematics/?p=926#h5p-1

THE FOOTNOTES

Notes

1. This rule is also called the GDS - Gross Debt Service Ratio.
2. There is also the Total Debt Service Ratio (TDS) that lenders use when borrowers have other loans. No more than 40% of a borrower's income can go towards their mortgage payment, property taxes, heating costs, half of their strata fees and other debt payments. They will take the minimum payment generated by the GDS (32%) and TDS rules.
3. Their payday could also land on the 3^{rd}, 17^{th} and 31^{st} of the month. This is more rare because only 7 months in the year have 31 days and February only has 28 days, normally. In general, there will be two months in the year where they have three paychecks.
4. The final year is the only exception. They will pay slighly less in the final year because the final payment in that year will be smaller.

5.12 Lump Sum Payments and Refinancing Mortgages

Learning Outcomes

Calculate the extra amount borrowed when refinancing a mortgage or the reduced payment size when renewing a mortgage and making a lump sum payment that drops the balance owing.

It is possible to borrow more money or pay off part of your mortgage when you go to renew your mortgage[1]. Let us first examine making a lump sum payment upon renewal (ie: making an additional payment when renewing your mortgage).

LUMP SUM PAYMENTS

After a mortgage term is up and before a borrower renews their mortgage, they can choose to pay down some of the balance owing with a **lump sum payment**. This is like the initial down payment but part-way through the mortgage. It is often advisable to make these payments, if you can afford them, as they will save you a lot of interest if there are many years left in your mortgage.

Example 5.12.1

The Frasers term is up on their mortgage. They owe $523,324.15 on their at the end of the term. They have $125,000 saved up to pay down on their mortgage before they renew the mortgage. They will renew for another 15 years. They negotiate an interest rate of 2.88%, compounded semi-annually. What is the size of their new monthly mortgage payments?

First, let us determine the amount they will borrow (PV):

$$\text{New PV} = \$523,324.15 - \$125,000 = \$398,324.15$$

Next, let's input the all values into the BAII Plus and calculate the size of their payments (PMT):

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	2	15×12=180	2.88	+398,324,15	**CPT** −2,724.56	0

The Frasers will pay $2,724.56 per month on their mortgage.

INCREASING THE SIZE OF A MORTGAGE UPON RENEWAL (REFINANCING)

After a mortgage term is up, a borrower can choose to increase the size of their new mortgage[2] by borrowing additional money against the mortgage when they renew. Unless necessary, it is often not advisable to increase the size a mortgage. This is because a borrower will pay a lot of interest on the extra money borrowed if there are many years left on the mortgage.

Example 5.12.2

Craig and Joel's mortgage term is up. For the last 5 years, their mortgage payments have been $3,904/month. Interest rates have dropped since they first bought their place. They are hoping to borrow some money to renovate their kitchen while still keeping their mortgage payments the same (at $3,904/month). How much can they borrow to renovate if they owe $523,324.15 and have 15 years left on their mortgage? Their new interest rate will be 1.99%, compounded semi-annually.

If we want to know how much Craig and Joel can borrow, we need to calculate the PV (amount owed) using the $3,904 payment and using the 15 years remaining to calculate N:

B/E	P/Y	C/Y	N	I/Y	PV	PMT	FV
END	12	2	15×12=180	1.99	**CPT** +607,464.81	−3,904	0

Take the difference between the amount they can borrow (PV) and the original amount they would have owed to determine how much extra Craig and Joel can borrow:

$$\text{Extra Amount Borrowed} = \$607,464.81 - \$523,324.15 = \$84,140.66$$

Conclusion: Craig and Joel can borrow $84,140.66 to renovate their kitchen[3].

YOUR OWN NOTES

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?

- These notes are for you only (they will not be stored anywhere)
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An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://pressbooks.bccampus.ca/businessmathematics/?p=940#h5p-1

THE FOOTNOTES

Notes

1. If you borrow more money when your mortgage term is up, this is called refinancing (instead of renewing).
2. Up to 80% of the value of the house can be financed (borrowed).
3. Provided that the new value of their mortgage does not exceed 80% of the assessed value of their house. Their house would need to worth at least $759,331.02 for the bank to lend them up to $84,140.66 for their renovation.

Chapter 5 Review Problems

1 Calculate the future value of the ordinary annuity in each part:

Size of Payment	Term of the Annuity	Nominal Interest Rate	Payment and Conversion Period	FV
a. $2,100	10 years	9.50%	6 months	?
b. $4.25	12 years	9.00%	1 day	?
c. $750	8 years	10.00%	1 month	?
d. $3,500	27 months	12.00%	3 months	?

Note: b. might explain how much an "average" smoker could save in 12 years (assuming a constant price for cigarettes and a fixed interest rate).

2 $500.00 is deposited at the end of every six months for nine years in an account paying 10.0% compounded semi-annually. Calculate the accumulated value of the deposits.

3 Calculate the amount of interest included in the accumulated value of $600.00 deposits made at the end of each month for 5 years. The interest rate is 13.5% compounded monthly.

4 Bill Holden is preparing retirement plans for his employees. He requires each employee to deposit $265.00 at the end of each month for 9 years. The interest rate is 8.75% compounded monthly.

a. How much money will be in each employee's account at the end of 9 years?
b. How much will each employee have actually contributed?
c. How much of the amount will be interest?

5 Corinne Smith made $2,750 deposits every 6 months into a registered retirement savings plan paying 11.25% compounded semi-annually. Just after making the 16th deposit, the interest rate changed to 10.00% compounded quarterly. If neither deposits nor withdrawals were made during the next five years how much would Ms. Smith then have in her account?

6 Calculate the present value of the ordinary annuity.

Size of Payment	**Term of the Annuity**	**Interest Rate**	**Payment and Conversion Period**	**PV**
a. $2,100	10 years	9.50%	6 months	?
b. $4.25	12 years	9.00%	1 day	?
c. $750	8 years	10.00%	1 month	?
d. $3,500	27 months	12.00%	3 months	?

7 You wish to take two years off work to attend school and also wish to receive $950.00 at the end of every month for the 2 years. If you were able to deposit money into an account paying 10.00% compounded monthly:

a. How much should be deposited when you take the time off?
b. How much interest will you receive in the two years?

8 A laptop was bought by paying $750 down and an installment contract with payments of $85 at the end of each month for 2.5 years. If the interest was calculated at 16.9% compounded monthly:

a. What was the equivalent cash price of the laptop?
b. How much was the cost of the financing?

9 Peter Van Dusen opened a trust account to fund his son's education. The account paid 10.25% compounded quarterly. His son is expected to require four years of quarterly payments of $2,000 with the first payment occurring 10 years 3 months from today . How much must Mr. Van Dusen deposit now so that his son will be able to receive the 4 years of payments? This is an example of a **deferred annuity.**

10 To purchase a new trawler-type yacht for chartering, Henry Skipper signed an agreement to borrow the entire amount and to make payments of $2,750 at the end of every month for seven years.

a. What was the purchase price of the yacht if money was worth 15.5% compounded monthly?
b. In his third year of operation, an economic downturn caused Mr. Skipper to miss payments 25 and 26. What payment was required at the time that payment 27 was due in order to bring the contract up to date?
c. Upon receipt of payment 27, the mortgage company wished to invoke a contractual clause and cancel the mortgage. How much (in addition to the payment calculated in part b above) would Mr. Skipper have to pay in order to fully pay out the mortgage?

11 Calculate the payment for each annuity:

Future Value	**Present Value**	**Interest Rate**	**Payment and Conversion Period**	**Time**	**PMT**
a. $0	$17,750	10.5%	1 quarter	4 years, 6 months	?
b. $12,000	$0	16.5%	6 months	20 years	?
c. $6,500	$0	8.4%	3 months	9 years, 3 months	?
d. $0	$12,500	10.5%	1 month	8 years	?

12 A used Corvette can be bought for $15,000 cash or for equal payments at the end of each quarter for 5 years. Calculate the size of the quarterly payments at 10% compounded quarterly.

13 A gaming computer priced at $4,600 can be purchased for $1,900 down and the balance paid by 36 equal monthly payments at 13.8% compounded monthly. Calculate the size of the monthly payments.

14 Calculate the term of each annuity:

Future Value	**Present Value**	**Size of Payment**	**Interest Rate**	**Payment and Conversion Period**	**Term (N)**
a. $0	$8,500	$675	10.75%	3 months	?
b. $23,500	$0	$491.25	8.25%	1 month	?
c. $0	$962	$100	10.00%	1 week	?
d. $85,000	$0	$2,500	13.00%	6 months	?

15 Charlie Horseshoe invested their $250,000 lottery winnings in a term deposit paying 8% compounded monthly for 10 years. For how long can $3,500 be withdrawn from the account at the end of each month starting at the end of the term deposit? Does your answer make sense? If not, why not?

16 $1,000 is deposited at the end of each month for 5 years. Find the nominal rate of interest compounded monthly at which the deposits will accumulate to $80,000. (Answer in % using 2 places of decimals.)

17 A camper van can be purchased for $27,500 plus 14% taxes and fees. The dealer will finance the balance owing after the payment of the sales tax and a down payment of 20% of the price (without the taxes and fees). Payments will be $330.30 at the end of each month for 7 years. What nominal rate of interest compounded monthly is being charged?

18 A computer valued at $2,800 can be bought for 25% down and monthly payments of $84.60 for 2.5 years. What **effective** rate of interest is being charged?

19 A loan of $30,000 is to be repaid with monthly payments over a period of 20 years. Calculate

the total savings in interest if the loan is financed at 12.25% compounded monthly rather than 13% compounded monthly.

[20] Judith Leisure-Lee is able to set aside $1,500 every 3 months from her income. She plans to buy a studio condominium at Hemlock Valley Ski Area when she has accumulated at least $50,000. How long will it take her if she can invest her savings at 9.5% converted quarterly? (State the answer in months.)

[21] A firm believer in technological education wishes to provide an educational institution with $6,000 bursaries to be awarded at the end of each year for the next 10 years. If the institution can invest money at 8.75% effective, how much should the philanthropist donate, one year prior to the first award, to set up the fund for the 10 bursaries?

[22] An agreement for sale contract carries payments of $4,500 at the end of every six months for 10 years. How much should you be willing to pay for the contract if you require a return of 12.25% compounded semi-annually on your money?

[23] Samuel Hardy received $48,650 as a severance settlement when his position was terminated. Harvey had just celebrated his 41st birthday. He immediately and prudently invested the money in an account paying a guaranteed 9.6% compounded semi-annually until his 60th birthday last year. At that time, he converted the existing balance into an ordinary annuity paying $3,750 per month with interest at 10% compounded monthly. For how long will the annuity run until all the funds have been paid out?

[24] Judith Leisure-Lee purchased a small studio condominium at Whistler for $115,000. She paid $40,000 down and agreed to make equal payments at the end of every month for 25 years. The interest rate was 13.25% compounded monthly.

a. What size payment is Judith making each month?
b. After 10 years of payments, how much will Judith still owe?
c. How much will she have paid, in total, over the 25 years?
d. How much total interest will she have paid after 25 years of payments?

25 A well-used car, priced at $1,250 was sold on "easy terms" for a down payment of $250 and $50 per month for two years. What effective interest rate is being charged?

26 Dogwood Holdings financed a factory expansion by borrowing $325,000 at 10% compounded semi-annually for 10 years. Payments are to be made at the end of every six months.

a. Calculate the size of the payment.
b. How much of the 4th payment is interest?
c. Calculate the outstanding balance after the 4th payment.
d. Construct an amortization schedule for the first 4 payments.

27 Save-On-Auto Parts borrowed $120,000 to purchase a fleet of seven vans. They intend to repay by making monthly payments of $2,400. Interest is at 16% compounded monthly.

a. How many full payments will Save-On-Auto Parts have to make?
b. Calculate the size of the final payment, to be made one month after the last full $2,400 payment, which will fully amortize the debt.
c. How much interest is included in the 40th payment?
d. What percentage of the loan will have been repaid by the first 48 monthly payments?
e. How much total interest will be paid?

28 A New York lottery offers a choice to the winner of $1,000,000 cash or $8,500 per month for 15 years. Which alternative should the winner select if money is worth:

a. 6.0% compounded monthly?
b. 6.25% compounded monthly?

29 Find the present value and future value for an annuity whose periodic payments of $1,000.00 are made at the beginning of every quarter for 7 years. The rate of interest is 13% compounded quarterly.

30 Find the present value and future value for an annuity whose periodic payments of $3,000 are made at the beginning of each six months for 15 years. The rate of interest is 8.5% compounded semi-annually.

31 Amby Dextrous bought a grand piano, and agreed to a series of monthly payments of $225 for 4 years starting the date of sale. The rate of interest is 13% compounded monthly.

a. Calculate the cash price.
b. How much will be paid over the term of its financing?
c. How much of the total payments will be interest?

32 Find the cash value of a three year service contract for monthly payments of $1,600 paid at the beginning of each month. The value of money is 14% compounded monthly.

33 If an annuity with a present value of $50,000 has periodic payments at the beginning of each quarter for 10 years, find the size of the periodic payment. The rate of interest is 12% compounded quarterly.

34 An annuity whose present value is $50,000 is extinguished by payments of $650.00 made at the beginning of each month for 12 years.

a. Find the nominal rate of interest compounded monthly.
b. Calculate the effective rate.

35 Payments of $200 were made into a stock ownership plan at the beginning of each quarter for 15 years. They now have a net value of $50,000. What has been the nominal rate of return on the investment?

36 $100 is deposited into a retirement plan at the beginning of every month for 20 years. One month after the last deposit, money is withdrawn in equal monthly payments for 15 years. If interest is 8.5% compounded monthly, find the size of the monthly withdrawals.

[37] If $75 is deposited into an education fund at the beginning of each month for 18 years and one month after the final deposit monthly withdrawals of $310.56 a month are made until the fund is exhausted, find the term of the annuity of withdrawals. Interest is 7.5% compounded monthly. (Try this problem both as an annuity due and an ordinary annuity.)

[38] Find the present value of a deferred annuity whose periodic payment is $550 at the beginning of each year for 20 years, with the first payment following a two year period of deferment. The interest rate is 3.6% compounded annually.

[39] Find the present value of a deferred annuity whose periodic payment of $360.00 is made at the end of every semi-annual period for 18 years after a deferment period of six months. The interest rate is 10% compounded semi-annually.

[40] A deferred annuity has a present value of $15,000 and periodic payments are made at the beginning of each quarter for 10 years after a deferral period of 8 years. The rate of interest is 6% compounded quarterly. Find the size of the periodic payment.

[41] Tri-City Holdings borrows $500,000 to fund the expansion of the firm into its eastern market region. If the loan is to be repaid by making equal payments at the end of each quarter for 8 years beginning after a deferral period of 2 years, find the size of the periodic payments. Interest rate is 11% compounded quarterly.

[42] Find the present value of a perpetuity whose periodic payments of $4,000 are made at the end of each quarter. Interest is at 6.8% compounded quarterly.

[43] An institute lecturer position in Math of Finance is being funded by a perpetual fund . The fund earns interest at I0% compounded annually and is to pay $60,000 at the end of each year, with the first payment two years from the date of the fund being set up. Find the size of initial funding required.

[44] You want to have accumulated $4,000 for your European trip four years from now. If interest is 6.4% compounded quarterly, find the size of quarterly deposits required for 4 years if deposits are made:

a. at the beginning of each quarter.
b. at the end of each quarter.

[45] A three-year car lease has a present value of $16,600. If money is worth 13.5% compounded monthly, find the equivalent monthly lease payments, payable in advance for 3 years.

[46] A Caribbean holiday tour package may be financed by making monthly payments of $300 at the beginning of each month for 2 years. Interest is 15% compounded monthly.

a. Find the purchase price now.
b. What will be the total paid in installments over the term?
c. How much interest will be paid over the term of the financing?
d.

[47] A building will produce net monthly incomes of $2,000 at the beginning of each month indefinitely. What is the maximum purchase price if one can get 14% compounded monthly on one's money?

[48] $400 is deposited into a retirement fund at the end of each quarter for 10 years and interest is paid at a rate of 9.6% compounded quarterly.

a. Find the accumulated balance at the end of the 10 years.
b. How much of that balance is interest?

[49] A recreational property is purchased for $54,000 with a down payment of 10% and the balance secured by a mortgage, amortized by equal monthly payments over 20 years. Interest is 16% compounded monthly.

a. Find the size of the monthly payments.
b. Find the balance after 5 years.
c. How much will have been paid for the property in total over the 20 year term of the financing?
d. How much of the total payments will represent interest?

50 An agreement for sale contract carries payments of $4,500 at the end of every six months for 10 years. How much should you be willing to pay for the contract if you require a return of 12.25% compounded **monthly** on your money?

51 John Leisure purchased a small studio condominium at Whistler for $115,000. He paid $40,000 down and agreed to make equal payments at the end of every month for 25 years. The interest rate was 8.75% compounded semi-annually. (see #36)

a. What size payment is John making each month?
b. After 10 years of payments, how much will John still owe?
c. How much will he have paid, in total, over the 25 years?
d. How much total interest will he have paid after 25 years of payments?

Simple Ordinary Annuities, Annuities Due, General Annuities

52 Starting today, Mrs. Robinson will put $500 into her RRSP every month for 20 years. If her RRSP earns 6% compounded monthly, how much will she earn in interest over the 20 years?

53 Mrs. Watson wants to save $52,450 for a down payment on a house. She will save $2,000 per quarter, **starting today.**

a. If her invested funds earn 6% compounded quarterly, how long will it take her to reach her goal?
b. How much interest will she earn during that time?

54 You have just graduated from BCIT. You owe $15,238.98 in student loans. You will be charged 7% interest compounded monthly. You can afford to make monthly payments of $300 starting today.

a. How long (in years) will it take you to repay your student loans?
b. How much interest will you have paid?

55 How long will it take to save $100,000 if you start saving$1,597 every 3 months, starting today? Assume an interest rate of 6%, compounded quarterly.

56 You borrow $50,000 from the bank to consolidate your credit card debt and student loans. The bank charges you 12% interest, compounded monthly. The first payment is one month from now.

a. How long will it take to pay off the loan if you pay$504.25 per month?
b. What is the cost of financing?
c. How long will it take to pay off the loan if you pay only $500 per month?

57 Mr. Eskanderian contributes $1,000 into his RRSP at the end of every quarter for 10 years. If his RRSP earns 10% compounded quarterly, how much interest will he earn in the 10 years?

Deferred Annuities

58 Judith transfers $25,000 into an RRSP today. She plans to let the RRSP accumulate earnings at the rate of 8.75% compounded annually for exactly 10 years and then immediately purchase a 15-year annuity. The first withdrawal will start 3 months after she purchases the annuity. The annuity earns 9% compounded quarterly. What size of payment will she receive every 3 months?

59 Katherine has recently received an inheritance. She wants to set aside part of the inheritance to put it into an RRSP to save for her retirement. She anticipates that she will need to receive $1,200 per month for 15 years with the first withdrawal starting exactly 10 years from today. The invested funds will earn 7% compounded monthly for the entire 25 years. What amount must she contribute to her RRSP today?

60 Barry wants to set up an annuity that will pay him $3,000 per month for 20 years beginning when he turns 65 years of age. If his current age is 50 years and the invested funds will earn 6.5% compounded monthly, what amount must he invest today?

61 Samuel recently inherited money from his grandfather's estate. He wants to purchase an annuity that will pay $5,000 every 3 months between age 60 (when he plans to retire) and age 65 (when his permanent pension will begin). The first withdrawal is to be 3 months after he reaches 60, and the last is to be on his 65th birthday. If Sam is currently 50.5 years old, and the invested funds will earn 6% compounded quarterly, what amount must he invest today?

62 It is time to start saving for your retirement. You are 40 years old and want to retire when you tum 60. You will deposit $1,000 into an RRSP at the beginning of every month for 20 years. You will then use the accumulated funds to purchase a 15-year annuity with the first withdrawal one month after your 60th birthday. Assume that the RRSP earns 7% compounded monthly, and the funds invested in the annuity earn 5% compounded monthly.

a. Find the size of the monthly withdrawals.
b. You decide that you will need at least $5,000 per month to live on when you retire at age 60. How much extra money must you contribute to your RRSP each month so you can withdraw $5,000 every month for 15 years?

63 Starting today, Giselle Lafleur will deposit $200 in her RRSP each month for 20 years. One month after the last deposit, she will withdraw the money in equal monthly withdrawals for 10 years.

a. Find the size of the monthly withdrawals if the invested funds earn j_{12} = 9%.
b. How much interest will Miss Lafleur earn over the next 30 years?

64 You have just celebrated your 20th birthday. You want to retire when you tum 65. Starting today , you are going to make monthly contributions to your RRSP, so that when you retire you can withdraw $2,250 per month for 15 years. The first withdrawal is made when you turn 65. You anticipate the invested funds will earn 7% compounded monthly.

a. How much money must you contribute each month to your RRSP to achieve your goal?
b. How much interest did you earn during the entire time?

65 Jean is thinking about retiring in five years. He would like to have $50,000 in an account when he retires. He decides to make monthly deposits **(at the beginning of each month)** in his local credit union where he can earn 7.0% **compounded quarterly.**

a. What would be the required monthly deposit to accumulate $50,000 in five years?
b. How much interest does John earn over the 5 years?

66 Six years from now, when you tum 55, you are planning to retire. You want to set aside

some money today so you can receive $2,500 at the end of every quarter for 15 years with the first withdrawal 3 months after you tum 55. The invested funds earn **9% compounded semi-annually.**

a. What amount must you invest today?
b. How much interest will you earn during the 21 years?

Perpetuities and General Annuities

[67] You have become wealthy beyond your wildest dreams and would like to create a scholarship at BCIT. You would like to give $2,000 per year to a student studying business math. You would like your scholarship to continue in perpetuity.

a. How much money should you set aside today if the first payment is in one year and the interest rate is 8% compounded annually?
b. How much should you set aside if the first scholarship is today?

[68] You take out a loan to buy a black Honda S2000 convertible sports car with leather interior . The bank requires that you put $9,000 down followed by payments of $925 at the end of every month for 5 years.

a. If the bank charges you 12% interest compounded monthly, what is the selling price of your car?
b. If the bank charges you 12% interest compounded **semi annually,** how much would your **monthly** payments be? **Note:** You borrow the same amount as found in part (a).

[69] Mr. Bean borrows $7,500 today. He will repay the loan with **quarterly payments** at the end of every three months for three years. The interest rate charged is 9% **compounded monthly.** Find the size of the payment.

[70] A corporation donates $11,000 to Langara. The funds are invested at 10% compounded annually per year. The interest is paid out each year as a scholarship. How much will be paid out each year if the first scholarship is paid immediately after the donation is received? **Note: It's easier to use the calculator.**

71 You have just won the Set for Eternity Lottery; the lottery will pay you and your descendants \$5,000 per month forever with the first payment in one month . Instead of receiving \$5,000 per month forever you would like to receive the cash now. What is the cash value of the prize if the interest rate is j_{12} = 6%?

72 The TRIUMF research lab at UBC recently received a donation from a private individual. The funds were matched two-to-one by the government. Each year they plan to pay out a scholarship of \$8,596.59 as they anticipate earning **11% compounded quarterly** on the invested funds. What amount did the private individual donate? (The first scholarship will be one year later). Round answer to the nearest dollar.

73 You have decided to purchase preferred shares of Plutonium Fuel Cells Inc. that pays a semi-annual dividend of \$1.25 per share.

1. What would you be willing to pay per share if you want to earn at least 5% compounded semi-annually on your investment and the next dividend is to be paid in 6 months?
2. You are short of cash and need to sell your shares of Plutonium Fuel Cells Inc. Unfortunately, interest rates have risen to 8% compounded semi-annually. Calculate your gain or loss per share if the next dividend is due in six months – and the dividend per share remains the same.

74 You purchase 500 preferred shares of Yardmucks Coffee that pays a quarterly dividend of \$0.26 per share. The next dividend is due in 3 months. Current interest rates are 8% compounded quarterly.

a. What would you be willing to pay for the 500 shares?
b. If interest rates drop to 6.5% compounded quarterly, how much would you expect to gain or lose if you sell all of your shares? Note: the next dividend is due in 3 months and the dividend per share is still \$0.26.

75 The Winfall lottery offers you two choices for its grand prize. Either a cash prize of \$1,500,000 today or \$7,000 per month forever (with the first installment one month from now).

a. Which choice should you select if interest is 6% compounded monthly?
b. You choose the \$1,500,000 today and deposit it into an account earning 6% compounded monthly. How much could you withdraw each month **forever?** Assume the first withdrawal is one month from now.

76 You borrow $50,000 and agree to make monthly payments for 15 years starting one month later. Calculate the size of your payments if the interest rate is 9% **effective.**

77 You borrow $20,000 today from a moneylender called, The Money Branch. As a result of your bad credit The Money Branch charges you an interest rate of 28.8%. You will repay the loan with equal monthly payments over four years with the first payment one month from now.

a. Find the size of the monthly payment if the interest is: (i)28.8% compounded monthly or (ii) 28.8% compounded semi-annually.
b. How much less interest would you pay if they compound the interest semi-annually, instead of monthly?

78 The University of Edmonton received a donation from a wealthy individual. Some of the donated money will be set aside to create a scholarship fund that will pay out $10,000 at the end of every 6 months, in perpetuity. If the invested funds can earn 8% compounded semi-annually, instead of 5% compounded semi-annually, **how much less money** must they set aside today to pay out a $10,000 scholarship?

79 You purchase 200 preferred shares of Black Bear Brewing that pays a dividend of $1 per share every 3 months. Current interest rates are 12% compounded monthly. The next dividend is due in 3 months.

a. How much would you be willing to pay for the 200 shares?
b. If interest rates fall to 8% compounded quarterly, how much would you expect to gain or lose if you sell all of your shares? **Note:** the next dividend is due in 3 months and the dividend per share is unchanged.

Mortgages

80 You take out a $200,000 mortgage:

a. What is the periodic interest rate per month if the rate is
    i. 12% compounded monthly?
    ii. 12% compounded semi-annually?

b. If you borrow $200,000, how much interest is paid in the first month? Use both rates and compare your answers.

81 A debt of $1,200 is repaid by monthly payments of $350 at the end of each month. The interest rate is 12% compounded monthly. Construct the complete amortization schedule **without** using the AMRT keys. Then go back and verify all the answers using the **AMRT** and **Pl/P2** keys .

Period	**Payment** **PMT**	**Interest** **INT**	**Principal Paid** **PRN**	**Balance Owing** **BAL**
0				
1				
2				
3				
4				
TOTALS				

82 You purchase a house in Surrey for $220,000 and put 25% down so you could receive a conventional mortgage. You decide to get your mortgage at Superstore since they give you free groceries each year as an incentive. A 5-year term mortgage (i.e., the interest rate is fixed for 5 years) is negotiated with Simplii Financial in which the balance is amortized over 20 years (repaid with equal payments over 20 years) at 6.70% interest compounded semi-annually.

a. Calculate your monthly payment. The bank rounds up the payment to the next dollar.
b. **COMP n** to verify that **n** is slightly smaller than 240. If the value of **n = -100** then you forgot to make the payment negative. If the value of **n is exactly 240** then you forgot to re-enter the payment.
c. Find out how the first payment is distributed between interest and principal. Compare this to the results for the 60th payment.
d. How much interest did you pay in the first year? By how much was the balance outstanding reduced in the first year?
e. What percent of the original mortgage was paid off in the first two years?
f. How much principal was paid off in the first five years? How much interest did you pay in the first five years?

g. How much interest did you pay in the fifth year of the mortgage? How much principal was repaid in the fifth year?
h. How much will you still owe after you have made five years of payments?
i. Calculate the value of the final payment assuming that the interest rate never changes during the 20 years.

83 The Archibald's are eligible for a Canada Mortgage and Housing Corp. insured mortgage allowing them to qualify for a mortgage of up to 95% of the selling price of the house. They are also subject to the 30% rule: no more than 30% of their gross income can go towards paying the mortgage and property taxes.

a. What is the maximum mortgage they qualify for if their gross monthly income is $5,000 and they want to amortize the mortgage over 25 years? Assume that the property taxes on the house are $1,800 per year (after the home owner's grant). Current mortgage rates are 6.80% compounded semi-annually. **Round answer to the nearest $100.**
b. The Archibald's take out a mortgage for $195,000 with Citizen's Bank amortized over 25 years at 6.8% interest compounded semi-annually for a 5-year term. What is the size of the monthly mortgage payment**(round up to the next dollar)?**
c. How much interest did they pay in the first five years of the mortgage?
d. How much money would they still owe on this mortgage after five years of payments?
e. When they renew their mortgage in five years time, mortgage rates have fallen to 6.0 % compounded semi annually for a five-year term. They have saved $15,000 and will use it to reduce the size of the mortgage. Find the size of their new monthly payments assuming the balance outstanding is amortized over the remaining time. The bank rounds up the payment to the next dollar.
f. What is the size of the final payment of the renewed mortgage assuming that the interest rate does not change during the remaining 20 years?

84 Banks normally use the 30% rule: no more than 30% of your gross income can go towards paying your mortgage and property taxes.

a. If your gross monthly income is $4,000 per month and your property taxes are $2,400 per year (after the $470 home owners grant), what is the largest mortgage a bank would authorize if the mortgage is amortized over 25 years and the rate of interest is 7.45% compounded semi-annually. **Round answer to the nearest $100.** (Assume monthly payments).
b. You purchase a fixer-upper house in downtown Mission for $164,000. Your down

payment is 25% and you negotiate a first mortgage for the balance. The mortgage is at 7.45%, compounded semi-annually, amortized over 25 years for a 3-year term. How large a monthly payment is required? The bank rounds payment up to the next dollar.

c. How much interest did you pay in the first three years of the mortgage?
d. What percent of the original mortgage have you paid off in the first three years?
e. How much interest did you pay in the third year only?
f. In three years the term of your mortgage is up and you wish to renew. Interest rates have increased to 9.25% compounded semi-annually for a three-year term mortgage. At this time you make a lump-sum payment of $10,000 to reduce the size of your mortgage. Calculate the size of the new monthly payments assuming the balance outstanding is amortized over the remaining time. Round up to the next dollar.
g. What is the size of the final payment of the renewed mortgage assuming that the interest rate does not change during the remaining 22 years?

[85] Suppose you take out a mortgage amortized over 25 years for $179,940 at j2 = 12.5%.

a. Find the size of the monthly payment. Round payment to **nearest penny.**
b. How much time and money would you save if you make 26 bi-weekly payments (equal to half of the monthly payment) instead of twelve monthly payments each year? Assume that the interest rate never changes during the 25 years.

[86] In March 2012 the Beckers purchased a house in Delta for $512,500. They made a down payment of exactly 20% and took out a mortgage with TD Canada Trust for the balance at an interest rate of 7.5% compounded semi-annually, for a 5-year term, amortized over 25 years.

a. What is the size of the monthly payment? The bank rounds the payment up to the next dollar.
b. How much of the 30th payment was interest?
c. How much interest did the Beckers pay in the 3rd year of the mortgage?
d. Today, (March 2017), they have decided to increase the size of their mortgage and use the money for house renovations. How much extra money can they borrow if they want to keep the same monthly payment as before but still pay off the mortgage by March 2037 (twenty years from now)? The interest rate has fallen to 5.1%, compounded semi-annually for a five-year term.
e. The Beckers increase their mortgage to $450,000 and amortize it over 20 years at 5.1%, compounded semi annually. What is the size of their monthly payment? (The

bank rounds up the payment to the next dollar.)

f. What is the size of the final payment assuming that the interest rate stays the same over the remaining twenty years?

[87] The Blacks are considering purchasing a three bedroom townhouse in the Killarney area of Vancouver. Their gross monthly income is $12,000 per month. The property taxes on the townhouse are $3,000 per year, and they also have to pay a monthly maintenance fee of $150 per month for the upkeep on the townhouse complex. Banks normally use the 30% rule: no more than 30% of your gross income can go towards paying your mortgage, property taxes, and monthly maintenance fees. What is the largest mortgage a bank would authorize if the mortgage is amortized over 25 years and the rate of interest is 5.6%, compounded semi-annually? (Assume monthly payments.)

[88] Five years ago the Smiths purchased a home in North Vancouver for $650,000. They made a down payment of exactly 20% and mortgaged the balance with Westminster Savings Credit Union. The interest rate was 5.6%, compounded semi-annually, for a 5-year term, amortized over 25 years.

a. Calculate the size of the monthly payment required. The credit union rounds the payment up to the next dollar.
b. What percentage of the original mortgage was paid off in the first 3 years?
c. How much interest did the Smiths pay in the fourth year of the mortgage?
d. The Smiths, after having made 5 years of payments, made a lump-sum payment to reduce the outstanding balance to $450,000. What was the amount of the lump-sum payment?
e. After making the lump sum payment, the Smiths renew their mortgage for another 5-year term, amortized over the remaining time, at 6.8%, compounded semi-annually. Calculate the new monthly payment. Round the payment up to the next dollar.
f. Assuming the interest rate remains the same over the remaining time, what is the size of the final payment?

[89] In October 2012 the Reids obtained a mortgage for $380,000 at 6.8% interest compounded semi-annually, for a 5-year term, amortized over 25 years. (The bank rounds the payment up to the next dollar.) Today, (October 2017), they have decided to increase the size of their mortgage and use the money for house renovations. How much more money can they borrow if they want to keep the same monthly payment as before but still pay off the mortgage by October 2037

(twenty years from now)? The interest rate has fallen to 4.3%, compounded semi-annually for a five-year term.

[90] You purchase a new car. The dealer offers you terms of 20% down and the remainder financed over five years at an interest rate of 8% compounded monthly.

a. Find the size of your monthly payment if your first payment is due at the end of the month and the price of the car was $33,906.25 including GST, PST, documentation fee, and government environmental levies.
b. What is the cost of financing (how much interest will you pay over the life of the loan)?

[91] You would like to save for your retirement by making monthly deposits of $200 into an account. What nominal interest rate of interest, compounded monthly, must you earn to accumulate $1,000,000 in thirty years? Assume that the first payment will be at the end of the month.

[92] You are considering quitting smoking due to the high cost of a pack of cigarettes. You smoke 1 pack a day at a cost of $7.50. If you put the $7.50 you would have spent on cigarettes, into a savings account earning 5.75% interest compounded daily, how much would you have in the bank at the end of ten years?

[93] Upon graduation, you have a student loan of $15,000. The most you can afford to pay is $550 per month. How long will it take you to repay the loan with payments of $550 per month starting in one month if the interest rate is 6% compounded monthly?

[94] Kent sold his car to Carolyn for $1,000 down and monthly payments of $120.03 at the end of every month for 3.5 years. The interest rate charged is 12%, compounded monthly. What was the selling price of the car?

[95] Rajinder bought a car with $5,000 down followed by equal monthly payments of $783.41 at the end of every month for 2 years at 16% compounded monthly.

a. What is the selling price of the car?

b. What is the cost of financing?

[96] Manpreet paid $14,000 to buy a used car. He made a down payment followed by equal monthly payments of $249.50 at the end of every month for 4 years at 8% compounded monthly.

a. What is the size of the down payment?
b. What is the cost of financing?

[97] For $42,000 an individual can purchase a 5-year annuity from Continental Life and receive monthly payments of $871.85 for 5 years with the first payment one month from now. What **effective** rate of interest does this investment earn?

Deferred Perpetuities

[98] A scholarship fund is to be set up. The fund will pay out a scholarship of $20,000 every year with first scholarship paid out two years after the fund is set up. Find the size of the donation needed. Assume the fund will earn 10% effective.

[99] A bursary fund for BCIT honor students is to be funded by a perpetual fund. The fund earns interest at 10% compounded annually and is to pay $30,000 each year, with the first payment four years after the fund is set up. Find the size of the initial funding that is required.

[100] An alumnus wants to donate a sum of money to his Alma Mater that will provide a scholarship of $750.00 every six (6) months in perpetuity. If money can be invested at 6% compounded semi-annually and the first $750.00 is to be awarded at the end of one year, how much must he donate to the school today?

[101] You are considering purchasing shares of New Wave Technology Corp. The company has stated that they will pay dividends of $0.72 per share every three (3) months with the **first dividend paid exactly four (4) years from today.** If the current interest rates are 8% compounded quarterly, what would you be willing to pay for one share **today?**

Using All 5 Keys

[102] You just celebrated your 40th birthday. You dream about retiring when you tum 55. You currently have $80,000 accumulated in your retirement plan. You decide to make deposits each month into a retirement plan for exactly 15 years, starting **today.** You want to purchase an annuity which will pay you $6,000 per month for l 0 year, with the first withdrawal starting one month after your 55th birthday. The retirement plan and the annuity earn 6% compounded monthly.

a. How much must you deposit each month into your retirement plan?
b. How much interest will you earn over the ENTIRE 24 years?

[103] Barney just celebrated his 40th birthday . He currently has $52,034 accumulated in his retirement plan and he plans to continue making equal monthly deposits into a savings account for 15 years, starting **today.** Two months after his last deposit, he intends to withdraw $4,000 per month for his living expenses for a period of 5 years. The invested funds earn 6%, compounded monthly, for the entire 20 years.

a. Find the size of the monthly deposits.
b. How much interest will he earn over the entire 20 years?

[104] You have $50,000 in your RRSP today. For the next 12.5 years you will contribute $500 per month into your RRSP with the first deposit made one month from now. How much will you have in your RRSP at the end of 12.5 years if you earn 6.9598% compounded monthly on your RRSP?

[105] You take out a loan for $50,000 and make monthly payments of $500 with the first payment made one month from now. If the interest rate on the loan is 6.9598% compounded monthly, how long (in months) will it take you to pay off the loan?

[106] Starting today, you will contribute $695.09 per month into an RRSP for a period for 5 years. You will then use the accumulated funds to purchase an annuity with monthly payments paid out over 12.5 years. What will be the size of the monthly withdrawals if the first withdrawal is made 2 months after the last deposit? Assume you earn 6.9598% *compounded monthly* the entire time.

107 What amount will be in an RRSP after 20 years if monthly contributions of $300 are made for the first 15 years and then contributions of $600 per month are made for the subsequent 5 years? The first deposit is made one month from now and the funds invested in the RRSP earn 7% compounded monthly.

108 Starting today, you contribute $1,200 every 3 months into your RRSP for five years. The interest rate was 10% compounded quarterly for the first 2 years and 9% compounded quarterly for the last 3 years. How much will you have in your RRSP at the end of 5 years?

109 Herb has made contributions of $2,000 to his RRSP at the end of every 6 months for the past 8 years. The RRSP has earned 9.5% compounded semi-annually. Today, he moved the funds to a different play, paying 8% compounded quarterly. He will now contribute $1,500 at the end of every 3 months. How much will he have in his RRSP 7 years from now?

110 The Alumni Association of BCIT would like to set up a scholarship that will pay out $2,000 **every 6 months** forever. The funds will be deposited in an account which earns 8.16% **effective.** The first scholarship will be awarded 2 years after the funds are deposited. How much money do they need to set aside today?

111 You plan to contribute $900 every **6 months** into your retirement plan for a period of 15 years. How much INTEREST would you earn over the 15 years if your RRSP earns 7% compounded **monthly?** The first contribution is made 6 months from now.

112 Barney just celebrated his 25th birthday. Starting **today** he will make contributions every **month** into his retirement plan for a period of 30 years. How much must Barney contribute every month to his retirement plan so he can withdraw $5,000 per month for a period of 20 years with the first withdrawal starting two months after his last deposit? Assume Barney earns 8.5%, compounded **quarterly,** the entire time.

113 Marika has already accumulated $18,000 in her RRSP . If she contributes $2,000 at the end of every 6 months for the next 10 years, and $300 per month for the subsequent 5 years, what amount will she have in her plan at the end of the 15 years? Assume that her plan earns 9%, compounded semi annually, for the first 10 years and 9% compounded monthly for the next 5 years.

[114] How much larger will the value of an RRSP be at the end of 25 years if the contributor makes month-end contributions of $300 instead of year-end contributions of $3,600? In both cases the RRSP earns 8.5%, compounded semi-annually.

[115] What will be the amount in an RRSP after 25 years if contributions of $2,000 are made at the beginning of each year for the first 10 years and contributions of $4,000 are made at the beginning of each year for the subsequent 15 years? Assume that the RRSP earns 8%, compounded quarterly.

Notes

1.
   a. $67,631.83
   b. $33,511.96
   c. $109,635.81
   d. $35,556.87
2. $14,066.19
3. $15,021.08
4.
   a. $43,307.02
   b. $28,620.00
   c. $14,687.02
5. $112,179.12
6.
   a. $26,734.40
   b. $11,382.03
   c. $49,426.12
   d. $27,251.38
7.
   a. $20,587.31
   b. $2,212.69
8. a. $2,818.12 b. $481.88
9. $9,443.94 if you assume the first withdrawal occurs 10 years and three (3) months from today *or* $9,685.94 if you assume that the first withdrawal occurs at the 10 year point.
10.
    a. $140,461.30
    b. $8,357.02
    c. $110,460.68
11.
    a. $1,250.01

b. $43.36
c. $117.93
d. $193.00

12. $962.21
13. $92.02
14. n = 15.58 → 16 quarters n = 41.5 → 42 months[n = 9.72 → 10 weeks n = 18.51 → 19 semi-annual periods
15. Payments are indefinite because the withdrawals are less than the interest accumulated for the period.
16. 11.2269% compounded monthly
17. 6.8381% compounded monthly
18. 15.2222% compounded monthly = 16.3305% effective
19. $3,816.00 total savings
20. n = 24.84 → 25 quarters or 75 months
21. $38,933.32
22. $51,094.94
23. 123 months at $3,750 and one (1) month at a smaller amount
24.
    a. $860.03
    b. $67,097.29
    c. $257,998 with the last payment= $849.03
    d. $257.998 − 75,000 = $182,998
25. 19.7469% effective
26.
    a. $26,078.85 (rounded up)
    b. $14,700.73
    c. $282,636.43

Pmt#	**Amount**	**Interest**	**Principal**	**Balance**
0				$325,000.00
1	$26,078.85	$16,250.00	$9,828.85	315,171.15
2	26,078.85	15,758.56	10,320.29	304,850.85
3	26,078.85	15,242.54	10,836.31	294,014.55
4	26,078.85	14,700.73	11,378.12	282,636.43

27.
    a. 82 full payments and one (1) smaller payment
    b. $2,266.47

c. $1,058.99
d. 44.4%
e. $79,066.47

28. a. PV of monthly payments= $1,007,279.88 which is more than the $1 million cash payment. Choose the payments.
b. PV of monthly payments= $991,342.82 which is less than the $1 million cash payment. Choose the $1 million cash payment.

29. PV = $18,794.90; FV = $46,021.60
30. PV = $52,476.38; FV = $182,913.49

31. a. $8,477.78
b. $10,800.00
c. $2,322.22

32. $47,360.41

33. a. $2,100.12
b. $10,800.00
c. $2,322.22

34. a. 11.9930%,
b. compounded monthly = 12.6747% effective

35. 16.1008% compounded quarterly
36. $617.43
37. n = 185.9986 → 186 monthly payments (15 years and 6 months)
38. $7,477.38 assuming that the first payment occurs at year two
39. $5,673.21 assuming that the first payment occurs at 12 months
40. $795.49 with first payment at year eight
41. $29,439.84 assuming that there are eight years of payments (for example, n = 32)
42. $235,294.12
43. $545,454.55

44. a. $217.86
b. $221.35

45. $557.06

46. a. $6,264.61
b. $7,200.00
c. $935.39

47. $173,428.57

48. a. $26,370.83
b. $10,370.83

49. a. $676.16 (rounded up)

b. $46,036.44
c. $162,261.81 with the last payment= $659.57
d. $162,261.81 − $48,600 = $113,661.81

50. $50,447.62

51. a. $608.72 (rounded up)
b. $61,465.87
c. $182,606.20 d.
d. $107,606.20

52. $112,175.55

53. $5.5 years, $8,450 interest earned

54. a. 5 years
b. $2,761.02

55. 11 years

56. a. 40 years and $192,040 of interest
b. b. You never will since the payment only covers the interest.

57. $27,402.55

58. $1,766.18

59. $66,820.18

60. $152,996.91

61. $48,752.43

62. a. $4,143.48
b. $1,206.71−$1,000 = $206.71 per month

63. a. $1,692.10
b. $1,692.10 ×120 − $200 × 240 = $155,052

64. a. $66/month
b. $2,250 × 180− $66 ×540 = $369,360

65. a. $695.09
b. $50,000 − $695.09 × 60 = $8,294.60

66. a. $48,559.18
b. $2,500 × 60 − $48,559.18 = $101,440.82

67. a. $25,000
b. $27,000

68. a. $50,583.41
b. $918.93 per month

69. $720.87

70. $1,000

71. $1,000,000
72. $25,000
73.
    a. $50.00/share
    b. a loss of $18.75/share
74.
    a. $6,500
    b. Gain of $1,500.
75.
    a. The PV of the $7,000 perpetual payment is $1,400,000, so $1,500,000 today is better
    b. $7,500 per month forever
76. $496.74 per month
77.
    a. i. $706.23/month ii. $688.03/month
    b. $873.60
78. $150,000 less
79.
    a. $6,600
    b. $3,400 gain
80.
    a. i. 1% per month ii. 0.975879417% per month
    b. $2,000 for the 1st month 1% per month and only $1951.76 using 0.975879417% per month.

81.

Period	**Payment PMT**	**Interest INT**	**Principal Paid-PRN**	**Balance Owing-BAL**
0				$1,200.00
1	$350	$12.00	$338.00	862.00
2	350	8.62	341.38	520.62
3	350	5.21	344.79	175.83
4	175.83 + 1.76 = $177.59	1.76	175.83	0
TOTAL		$27.59	$1,200.00	

Final payment is $350−$172.42 = $177.58 using the calculator (slight difference due to rounding).

82.
    a. 1240.74 → $1241.00/month
    b. n = 239.8969885
    c. 1st month: $332.35 principal, $908.65 interest 60th month: $459.53 principal, $781.47 interest
    d. Principal paid off= $4,111.26: Interest is $10,780.74

e. e. $8,502.60/ $165,000 = 5.153%
f. Principal paid off= $23,554.39: Interest is $50,905.61
g. Principal paid off= $5,351.30: Interest is $9,540.70
h. Would need $141,445.61 to pay it all off
i. $1,113.48

83.
a. $196,200 rounded
b. $1,341.82 → $1,342/month
c. $62,591.48
d. $177,071.48
e. $1,154.254 - $1,155/month
f. $813.47

84.
a. 137,283.97..., $137,300 (rounded)
b. $895.95 ..., $896/month
c. $26,478.31
d. $5,777.69/$123,000 = 4.697%
e. $8,683.64
f. 939.527 ..., $940/month
g. The balance outstanding is $107,222.31
h. $545.58

85.
a. $1,920.00 per month
b. 447.8742287/26 = 17.2259 years, so save 25 years - 17.23 years = 7.77 years The interest savings is: $1,920 × 300- $960 × 447.8742287 = $146,041

86.
a. $3,000 (n = 299.8201542)
b. $2,430.53
c. $29,143.82
d. $77,273.24 ($452,806.23 −$375,532.99)
e. $2,982
f. $2,737.26

87. $519,290.79, $519,300

88. a.
a. $3,205 (n = 299.8729184)
b. 6.056% ($31,492.35/$520,000)
c. $26,748.44
d. $14,419.50
e. $3,410

f. $3,297.03

89. $76,785.12 ($421,860.24- $345,075.12)

90. a. $550
    b. $5875

91. 13.58942%

92. $36,994.34

93. 30 months

94. $5,100

95. a. $21,000
    b. $2,801.84

96. a. $3,780
    b. $1,756

97. j_2 = 9.38064%

98. $181,818.18

99. $225,394.44

100. $24,271.84

101. $26.75/share

102. a. $1,177.37
     b. $428,073.40

103. a) $271.00/month b) $139,186 53

104. $238,086.11

105. 150 months or 12.5 years

106. $500/month

107. $177,755.87

108. $30,723.99

109. $136,300.67

110. $44,449.82

111. $19,855.69

112. $352.39/month

113. $188,830.07

114. Month-end: $302,244.75 Year-end: $290,846.96 $11,397.79 more

115. $223,904.53

Chapter 6: Investment Decisions

An investment consists of spending money or other resources (e.g., time) with the aim of receiving benefits later.

INVESTMENT → BENEFITS

(now) (later)

Depositing money in an interest-bearing account is an **investment**. The purchase of a bond or treasury bill is an investment. Also, company projects such as the purchase and installation of equipment, the development of new products and increasing inventory held for sale are investments.

In this chapter we consider the problem of deciding whether or not an investment is worthwhile and deciding which investments are preferable to others.

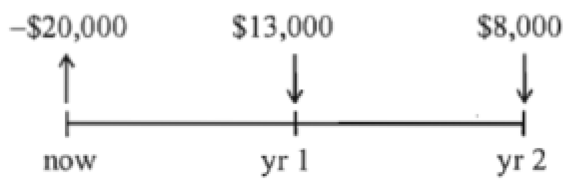
We view the financial aspect of an investment as being *entirely determined by the cash flows* associated with it by **how much** money is spent and received and **when** it is spent and received.

To (substantially) simplify the analysis we will omit tax considerations and view the expected cash flows of projects as being certain. Our methods and calculations will be similar to cases with taxes and risks included, but will be less complex.

6.1 Evaluating a Business Plan

Example 6.1.1

Suppose a business plan would require an investment of \$20,000 now and would result in cash flows of \$13,000 in one year and \$8,000 in two years.



Then, if you ignore (just for a moment!) the time value of money, you can see that:

$$\begin{aligned} Profit &= Revenue - Expense \\ &= \$21,000 - \$20,000 \\ &= \$1,000 \end{aligned}$$

showing a positive profit.

Key Takeaways

A positive profit is a minimum requirement for a successful investment. No company would choose investments which they expect to result in a loss.

Beyond this the question is: How can we decide whether this investment is preferable to others?

Suppose that an alternative investment was to place the money in a trust account which would pay 6% effective over the two-year period, and from which withdrawals could be made at any time.

Then, if you try to match the sequence of cash flows you would have:

Time	**Interest**	**Deposit**	**Withdrawal**	**Balance**
0		$20,000.00	$0.00	$20,000.00
1	$1,200.00		13,000.00	8,200.00
2	492.00		8,000.00	692.00

The final balance of $692.00 shows that in the account we could have withdrawn a total payment of $8,692 at the end of the second year, which is more than the original investment would allow; hence, the trust company account would be a better investment. The first investment made money, but **not fast enough** to earn 6% effective.

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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6.2 Rate of Return

The rate at which profit is earned, the *rate of return*, is the key factor in investments. It is described and used in **exactly the same way as a compound interest rate.** Usually it is stated as an *annual rate of return* (a rate compounded annually, an effective rate). As in the usual calculation of compound interest, the rate of return is applied to the outstanding balance (the unrecovered part of the investment). Once the investment has been recovered, all additional inflows represent profit beyond the minimum requirements.

We need a way to **evaluate** investments so that we can choose the best ones. The major methods of evaluation involve discounting cash flows to find their present values (values at the beginning of the investment).

There are several approaches to dealing with the rate of return. The first approach we shall consider is to set a minimum rate, called the ***minimum acceptable rate of return* (MARR)**, or minimum ***return on investment* (ROI)**.

If a company sets such a rate, then investments must meet or exceed this rate of return before the company will fund them. Consequently, the company's resources will be directed to only those projects which provide adequate returns.

The setting of the minimum acceptable rate of return is a management decision and depends on a number of factors, including the company's historical earnings rate and the cost of capital raised by the company.

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6.3 Net Present Value

Suppose that we set a minimum rate of return of 6% effective and apply it to our first example. Then, the future inflows would have present values of:

$$\frac{\$13,000}{1.06} = \$12,264.15$$

And

$$\frac{\$8,000}{1.06^2} = \$7,119.97$$

So \$12,264.15 is the amount it is worth investing now to receive \$13,000.00 in one year (and earn 6%), and \$7,119.97 is the amount it is worth investing now to receive \$8,000.00 in two years (and earn 6% compounded annually).

Thus, to earn a rate of return of 6% per year, it would be worth paying a total of:

$$\$12,264.15 + \$7,119.97 = \$19,384.12$$

But the investment costs \$20,000, which is \$615.88 more than it should if 6% is to be earned! Consequently, the investment should not be made if 6% is the minimum acceptable rate.

The discounting procedure above finds the ***net present value* (NPV)** of investments. This dollar amount is the difference between what "should" be worth investing and what actually has to be invested. So the inflows (benefits) are discounted to see the maximum amount worth investing, and the outflows (costs) are discounted to see what actually is being invested.

Using PV for present value:

$$NPV = PV(inflows) - PV(outflows)$$

Example 6.3.1

A small manufacturing business will cost \$200,000 paid immediately and other expenses incurred are expected to result in a net negative cash flow of \$40,000 by the end of the first year. It is then expected to earn annual positive net cash flows of \$50,000 starting at the end of the second year and continuing each year until the end of the fifth year. It **will** be sold at the end of the last year (year five) for \$250,000.

If the investors have set a minimum rate of return of 20%, should they invest in the business?

To evaluate this investment by NPV we find for outflows:

PV of $200,000	is	$200,000.00
PV of $40,000 (yr 1)	is	$33,333.33
Total PV Outflows		**$233,333.33**
PV of $50,000 (yr 2)	is	$34,722.22
PV of $50,000 (yr 3)	is	$28,935.19
PV of $50,000 (yr 4)	is	$24,112.65
PV of $300,000 (yr 5)	is	$120,563.27
Total PV Inflows		**$208,333.33**

So that:

$$\begin{aligned} NPV &= PV(inflows) - PV(outflows) \\ &= \$208,33.33 - \$233,333.33 \\ &= -\$25,000 \end{aligned}$$

The negative NPV shows that the investment did **not** earn the required rate of return.

Note that the net outflow of $40,000 in year 1 was also discounted to its present value of $33,333.33 to make the evaluation. You could assume that if $33,333.33 were invested elsewhere it would earn 20% and be available as $40,000 at year 1.

To summarize decisions based on NPV:

- If, for a given required rate of return, an investment has **NPV > 0** (i.e., a positive net present value), then the investment earns **more** than the required rate.
- If **NPV < 0** (i.e., a negative net present value), then the investment earns **less** than the required rate.
- If **NPV = 0,** then the investment earns **exactly** the required rate.

Be careful to note that a negative NPV does not necessarily imply that the investment resulted in a loss. It may be that the investment earned a profit, but did not do so fast enough to make the required rate of return. This was the case in the example above, which would have had a profit of $210,000, but would not have made the profit at a rate of 20% effective.

Knowledge Check 6.1

To check the result of the last example, complete the following "account" calculation at 20% effective.

Time	Interest	Deposit	Withdrawal	Balance
0		$200,000		$200,000
1	$40,000	40,000		?
2	?		50,000	?
3	?		50,000	?
4	?		50,000	?
5	?		300,000	?

You should find a final balance of $62,208 which shows that to earn 20% an extra $62,208 would be required at the end of the fifth year. This again shows that the original investment did not earn 20%.

Question: How is the required $62,208 related to the NPV of -$25,000?

Knowledge Check 6.2

Find the present value of each cash flow in the above problem using a minimum acceptable rate of return of 15% effective, and use the NPV to determine whether or not the investment would be acceptable using this lower rate.

Now consider the problem of choosing between two different investments, which might both be acceptable when considered separately.Suppose the investors in Example 2, who are considering the manufacturing business, have also the choice of purchasing a share in an existing business, which is described below. If they have set a minimum rate of return of 15%, how can we determine which investment is preferable?

Example 6.3.2

The share of the existing business would cost $225,000 cash and is expected to produce cash inflows of $15,000 a year for five years, then to be sold for $425,000.

For the share investment we have the following cash diagram:

To evaluate this investment by NPV at 15% effective, we have for the only outflow a present value of $225,000, and for the inflows:

Time (year)	Inflow	PV
1	$15,000	$13,043.48
2	15,000	11,342.15
3	15,000	9,862.74
4	15,000	8,576.30
5	440,000	218,757.76
	Total PV Inflows	**$261,582.43**

Then the net present value is:

$$\begin{aligned} NPV &= \$261,582.43 - \$225,000 \\ &= \$36,582.43 \end{aligned}$$

In Knowledge Check 6.2 you should have found that at 15% the NPV of the manufacturing investment is $13,641.07; hence, by this criterion the purchase of the share would be more profitable.

Knowledge Check 6.3

Use the net present value criterion to evaluate the following investments:

OPTION A: A small restaurant business in leased premises can be purchased for $30,000. It is expected to produce cash inflows from operations of $12,000 a year (at the end of each year) for 4 years. At the end of the 4 years the lease will expire and the equipment will be sold for $14,000.

OPTION B: A similar restaurant with its own premises can be purchased for $92,000. It would produce net cash inflows of $21,000 at the end of each year for 5 years and would be sold for $95,000 at that time.

If the prospective investors set a minimum rate of return of 20%, should they purchase a restaurant, and if so, which one would be the best deal?

You should attempt the problem by discounting each cash flow and finding the NPV that way. In the following setup for OPTION A, we have assumed that outflows will be treated as negative. This is a standard practice and allows a straightforward total to get NPV.

Time (yr)	Cash Flow	PV
0	-$30,000	?
1	12,000	?
2	12,000	?
3	12,000	?
4	26,000	?
	(12,000 + 14,000)	
	NPV =	**$7,816.36**

For OPTION B you should get an NPV of $8,981.22.

On the basis of NPV both restaurants are acceptable investments, but the second restaurant (OPTION B**)** would be the most beneficial if only one is to be purchased.

Calculating the present value of the inflows can always be done by discounting each flow separately, but in the case of equal flows it can also be done by an annuity. For example, in OPTION A the first three cash flows of $12,000 can be treated as an annuity by setting:

$$n = 3;\ i = 20$$

and finding PV. Then find the present value of the final $26,000 at year 4. You should get a total of $37,813.36, which agrees with the result above.

Your Own Notes

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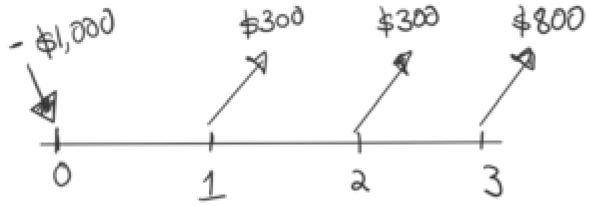
6.4 Cash Flow on the BAII Plus

Example 6.4.1

You invested $1,000 and received $300 for the first 2 years at the end of each year. At the end of the 3rd year, you sold your investment for $800. The minimum Annual Rate of Return (MARR) is 14%.



The NPV is:

$$NPV = -\$1,000 + \frac{\$300}{(1.14)^1} + \frac{\$300}{(1.14)^2} + \frac{\$800}{(1.14)^3}$$

But we will solve using the BAII Plus Calculator:

Step	**To**	**Press**	**Display**
1	Select Cash Flow worksheet	[CF]	CFo= (previous entered value)
2	Clear previous work	[2ND] [CE|C]	CFo= 0
3	Enter initial investment, negative for outflows.	[1][0][0][0][+\-][ENTER]	CFo = – 1,000
4	Enter 1st cash flow	[↓][3][0][0][ENTER]	C01= 300
5	Enter frequency of 1st cash flow (frequency =2)	[↓][2][ENTER]	F01= 2
6	Enter 2nd cash flow	[↓][8][0][0][ENTER]	C02= 800
7	Enter frequency of 2nd cash flow	[↓][1][ENTER]	F02= 1
8	To find NPV	[NPV]	I = 0
9	Enter the discount rate	[1][4][ENTER]	I = 14
10	Compute NPV	[↓][CPT]	NPV = 33.97536624
11	To find IRR	[IRR]	IRR = 0
12	Compute IRR	[CPT]	IRR = 15.69595604

We will write this as:

CF0 = -1000	
C01 = 300	F01 = 2
C02 = 800	F02 = 1

Your Own Notes

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6.5 Internal Rate of Return

When NPV is positive we know that the project being examined earns more than the required rate. When NPV is negative we know that the project being examined earns less than the required rate.

Only when NPV is zero does the project actually earn exactly the required rate. In this special case we can view the inflows as paying back our investment plus interest at the required rate. The *internal rate of return* (IRR) is the rate at which this happens, i.e., the rate for which:

$$\text{PV Inflows} = \text{PV Outflows}$$
$$NPV = 0$$

Consider the manufacturing company in Example 6.3.1. In that case at a 20% effective rate of return the present value of inflows was

\$208,333.33 and the present value of outflows was \$233,333.33 so that:

$$NPV = \$208,333.33 - \$233,333.33 = -\$25,000.00$$

We also found in Learning Activity #2 that the NPV at 15% was \$13,641.07.

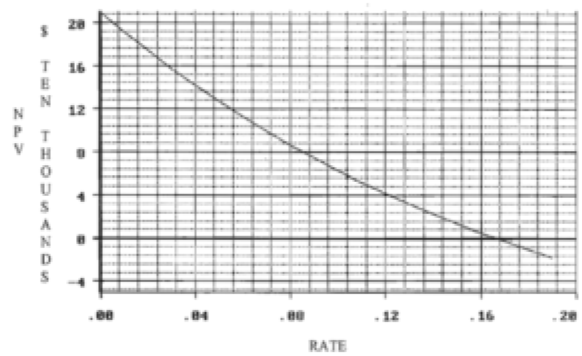
If the required rate is changed to 10% effective, the present value of inflows becomes \$299,315.12 and the present value of outflows becomes \$236,363.64 so that:

$$NPV = \$299,315.12 - \$236,363.64 = \$62,951.48$$

Similarly we can establish the following values:

Required Rate	16%	17%
NPV	\$5,156.73	-\$2,927.98

It should be clear that the rate which would make NPV = 0 is between 16% and 17% effective. From the graph below we can see that this value, the **internal rate of return** of this project, is about 16.6%.



The internal rate of return can be obtained from business calculators. In the case of the BAII Plus, place the cash flows in the calculator as for the NPV, remembering to clear all ([2ND][CF][2ND][CE|C]) before entering:

CF0 = -200,000	
C01 = -40,000	F01 = 1
C02 = 50,000	F02 = 3
C03 = 300,000	F03 = 1

and then pressing [IRR][CPT]

You should see the answer, IRR= 16.63227%

Note that this is a solution to the equation:

$$\text{PV Inflows} = \text{PV Outflows}$$
$$\frac{\$50,000}{(1+i)^2} + \frac{\$50,000}{(1+i)^3} + \frac{\$50,000}{(1+i)^4} + \frac{\$300,000}{(1+i)^5} = \$200,000 + \frac{\$400,000}{(1+i)^1}$$

Which is quite difficult to solve by hand.

The answer in your calculator will always be a **periodic rate** for whatever period was used for the flows. Since the cash flows were yearly, the rate is an effective rate in this case. Since we are working with estimates, it is not uncommon to work exclusively with annual rates, as we do in this textbook.

Your Own Notes

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6.6 Payback

The *payback period* of an investment is the length of time it takes to recover the initial investment. The shorter the time needed, the better the investment by this criterion. Unless stated otherwise, the time will be calculated using the undiscounted cash flows.

Example 6.6.1

Suppose that an investment of $20,000 will produce cash inflows of $2,000 in the first year and $6,000 annually for many years after that. Then the payback period of this investment is four years, since by accumulating the inflows you can obtain the following table:

Time (yr)	Cash Flow	Balance
0	-$20,000	-$20,000
1	2,000	-18,000
2	6,000	-12,000
3	6,000	-6,000
4	6,000	0

The payback is at four years.

Example 6.6.2

What if the final cash flow in year four was $10,000? Then we would have the following:

Time (yr)	Cash Flow	Balance
0	-$20,000	-$20,000
1	2,000	-18,000
2	6,000	-12,000
3	6,000	-6,000
4	10,000	4,000

In this case, convention says that we treat all inflows and outflows as taking place uniformly throughout the year, so payback is:

$$\begin{aligned}\text{Payback} &= \frac{\text{-Balance at that year}}{\text{Cash flow at the next year}} \\ &= \frac{\$6,000}{\$10,000} \\ &= 3.6 \text{ years}\end{aligned}$$

Note that no allowance is made for the exact timing of flows (if the inflows were $5,000 a year the same result would have been obtained) or for the size of cash flows after the payback period. Also, there is no mention of a **rate of return,** although a short payback period usually indicates a high rate of return.

If the investment is not paid back exactly at the end of a year, and, if the cash flows could reasonably be assumed to occur uniformly throughout the year, then you can estimate the payback period by adding the fraction of the last year needed to recover the investment. This is an approximation, but it is a useful tool.

Suppose that in Example 6.8 above the investment required had been $21,500. Then the payback period would have been extended to 4.25 years since the additional $1,500 would have to be earned in the 5th year and would require 1,500/6,000 = 0.25 years, so that the payback period would be 4 + 0.25 = 4.25 years.

If the cash flows were given on a monthly basis there would be no need to estimate a part of a month – the time could be given to the end of the first month at which the recovery is complete.

The payback period is one of the simplest measures of investment effectiveness. Its benefits are that it is easily understood, easily calculated and that it provides some assessment of the risk to which a company is exposed in an investment. Investments which require a long period before the investment is recovered are generally felt to carry a higher risk, since predictions of cash flows are more likely to be in error if they are for times far in the future.

Knowledge Check 6.8

Complete the tables and calculations below to obtain the payback periods for the restaurant OPTIONS A and B given in Example 2.

OPTION A ($30,000 invested at time 0)

Time	Inflow	Balance
0	-$30,0000	
1	12,000	
2	12,000	
3	12,000	

Time needed beyond end of second year:

Amount needed = ?

Amount in year three ?

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
- These notes are for you only (they will not be stored anywhere)
- Make sure to download them at the end to use as a reference

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6.7 Effects of Different Lifetimes

When we evaluate two options using the NPV criterion, we need to be mindful of the different lifetimes. A car with an NPV of $25,000 seems like a better investment than a car with an NPV of $30,000, but if the second car will last 10 years, but the first only five, that will change the calculation.

There are methods of evaluating investments with differing lifespans, such as the Equivalent Uniform Annualized Cost (EUAC) or Capitalized Cost, which give an average cost per year. We cover these optional topics in Appendix B.

6.8 Cost Comparisons

All of the methods presented in this chapter try to answer the question: Should we invest?

If there was a clear answer to this question that we could teach you, the world of business would be much simpler. Every time we need to make a decision, we will have ot evaluate the criteria given here, as well as looking at the risks and rewards. The concept of risk will be covered more thoroughly in Business Statistics.

Here are the tools we have covered:

1. PROFIT (Revenue – Costs) – if we do not end up with a profit, than it is useless to evaluate the NPV or IRR.
2. PAYBACK: How long until we get our original investment back? When we choose, a shorter payback is preferred.
3. NET PRESENT VALUE (NPV): When we evaluate the Profit with interest, is it positive?
4. INTERNAL RATE OF RETURN (IRR): What is our actual rate of return?

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
- These notes are for you only (they will not be stored anywhere)
- Make sure to download them at the end to use as a reference

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https://pressbooks.bccampus.ca/businessmathematics/?p=841#h5p-1

Chapter 6 Review Problems

[1] For an investment of $15,000 now, a company can receive $9,250 one year from now, and $9,200 two years from now. Check its earnings rate on this deal by comparing it to an account using the following rates. For each part, fill out the following balance sheet

Time (year)	Interest	Deposit	Withdrawal	Balance
0		$15,000		$15,000
1	?		$9,250	?
2	?		9,200	?

a. 10% effective
b. 20% effective
c. 15% effective

[2] For the investment in the previous problem , find the NPV at each of:

a. 10% effective
b. 20% effective
c. 15% effective.

Relate each value to the results in (a), (b) and (c)

[3] A finance company has an opportunity to purchase three promissory notes from a broker for a total of $140,000 cash. The first note would pay $20,000 in one year, the second would pay $70,000 in two years and the third $80,000 in three years. The finance company has set for itself a minimum acceptable rate of return of 10% effective.

a. Evaluate the net present value of the investment at the company's MARR.
b. State your conclusion about the acceptability of the profitability of this deal.

[4] A project is to cost $60,000 immediately and to produce net cash inflows of $20,000 at the end of the first year, $30,000 at the end of the second year and $25,000 at the end of the third

year. At the end of the third year the business will be sold for $5,000. The company aims at a rate of return of 15% effective.

a. Show whether or not the company will achieve its objective.
b. Find the value of the bonus paid at the start which would cause the project to earn exactly 20% effective.

[5] Samuels Co. plans to start a repair business. It would provide net returns of $35,000 at the end of the first year, and $50,000 at the end of each of the next three years. The equipment would then be sold for $11,000 and the business terminated. Samuels aims at a rate of return of 15%. How much should Samuels be willing to invest (now)?

[6] Williams Co. has the chance to start a transportation company in the north. It would require $108,000 to start the business and it would provide net returns of $35,000 at the end of the first year, and $40,000 at the end of each of the next two years. The equipment would then be sold for $9,000 and the business terminated. Williams aims at a rate of return of 12%.

a. Find the NPV at 12% effective, and the IRR. Should Williams Co. start this company?
b. If the government wanted to subsidize Williams so that it would earn **15%** compounded annually, what subsidy paid at the **end** of the three years would be required?

[7] PA Lumber Co. is bidding on the right to cut lumber from a private forest for 10 years. The amount of lumber permitted to be cut each year would allow PA to produce a net cash flow (excluding cost of cutting rights) of $120,000 a year. If PA aims at a rate of return of 12.5%, how much could it afford to bid as a lump sum for the cutting rights?

[8] Iffi Co. plans to install insulation in a building which it plans to use for six years. The insulation and its installation will cost $193,000 and will result in savings of $37,000 a year and will increase the residual value of the building by $90,000. Find the internal rate of return of the planned installation of the insulation.

[9] Tessa Quaid has purchased a 10-year video rental franchise for $12,000. She will have to invest another $22,000 in the store immediately. She expects returns of $8,000 a year at the end

of each year. Will TQ make a rate of return of at least 15%? What will be the internal rate of return?

[10] TW Cable Co. plans to begin operation in a large town and to expand later to a nearby small town. TW estimates its net cash flows to be:

Time (year)	**Cash Flow**
0	-$600,000
1	100,000
2	100,000
3	-50,000 (expansion)
4-9	150,000
10	700,000 (includes sale of business)

Find the net present value at 17% and the internal rate of return.

[11] A food concession at an airport has a five year life and costs $130,000. Initial investments in equipment, training and inventory amount to $75,000. The concession's operation is expected to produce net cash inflows of $50,000 a year and the residual value of equipment and the ending inventory are expected to total $30,000.

a. Find the NPV at 15% effective.
b. Find the internal rate of return.

[12] The owners of a small business are considering three offers from potential purchasers:

- Offer #1: $100,000 cash.
- Offer #2: $80,000 now and $10,000 at the end of each year for five years.
- Offer #3: $25,000 now and $25,000 at the end of each year for five years.

Which offer is most attractive if money is worth 9% to the owners?

[13] A "10%" Government bond is to pay interest of $100 per year for nine years. Find the price investors should be willing to pay for the bond if they want to earn:

a. 15% effective.
b. 10% effective.
c. 5% effective.

[14] The Titan Package Co. finds that a $95,000 investment in automated packaging equipment will save it $20,000 a year for the next 10 years. The equipment will have no value at the end of the 10 years. If the company aims at a rate of return of 15% effective should it purchase the equipment? Why/why not?

[15] The Fast Food Co. can expand into either of two locations. Location A will need an expenditure of $130,000 now and will result in a yearly net cash inflow of $35,000 (at the end of each year) for 10 years and no other benefits. Location B will need expenditures of $150,000 now and $50,000 in one year. It will then produce a yearly net cash flow of $45,000 a year for the following 12 years (all at the end of each year). In location B there would also be no other benefits. Which location gives the higher net present value at a required rate of return of 16% effective per year?

[16] You are considering a home-based business. You would like to open a kennel. You estimate that your start-up costs will be $250,000 this year and you will need to spend another $10,000 next year. You expect revenues of $35,000 per year for the first two years and $42,000 for the next three years. Your expenses are thought to be $6,000 per year. At the end of five years you plan to sell the kennel and anticipate receiving $255,000. You want to earn a minimum of 14% compounded annually on your investment. All revenue occurs at the end of the year and all expenses are paid at the beginning of the year.

a. Calculate the payback for this project.
b. Calculate the net present value for this project. Should you open a kennel?
c. If you require only 12% effective should you open a kennel? Why or why not?
d. How much should the start-up costs be reduced by so that you would be willing to open a kennel? Use a MARR of 14%.
e. Your estimate of the kennel's selling price may be too low. You think you may be able to sell the kennel for more than $255,000. What is the minimum selling price you must have so that you would be willing to open the kennel? Assume the start-up costs remain unchanged. Round to the nearest $. Use a MARR of 14%.
f. Calculate the internal rate of return (IRR) for this project. Should you open a kennel if you want to earn at least 14% compounded annually on your investment?

17 You are considering purchasing a home-based business that uses AI to create customized meal plans. You think that you can buy the business from the current owner for $120,000. You expect revenues of $40,000 per year. Your expenses are thought to be $20,000 per year for the first 2 years and $10,000 per year for the last 3 years. At the end of five years you plan to sell the business, retire and move to Las Vegas. You anticipate you could sell it for $130,000. You want to earn at least 16% compounded annually on your investment?

a. Calculate the payback for this project.
b. Find the NPV. Should you buy the business?
c. How high could the purchase price of the business be and still make it worthwhile for you to buy the AI business?
d. You think you might be overly optimistic in your estimate of the selling price of the business. What is the lowest selling price that you can accept and still be willing to buy the business? Round to the nearest $.
e. Calculate the internal rate of return (IRR) for this investment. Should you buy the business?
f. Your accountant tells you that your projected annual revenue is too high. What is the maximum annual decrease in revenue you could withstand and still have this investment be worthwhile?

18 You want to start up a business in a mall selling CFL souvenirs. You plan to spend $80,000 in start up costs. You anticipate that you will break even in the first year and second year, make $20,000 the third, and make $20,000 each year after that. You plan to sell the business for $100,000 at the end of the sixth year. You want to earn at least 15% effective on your investment.

a. What is the net present value of your business plan?
b. What is the internal rate of return?
c. Should you undertake the business plan? Why or why not?
d. You think you might be overly optimistic in your estimate of the selling price of the business. What is the lowest selling price that you can accept and still undertake the project? Round to the nearest dollar.

19 You are a restaurant owner and are considering expanding your business by buying the recently vacated store next door. The purchase price is $50,000. You will also need to spend an extra $35,000 in re-modeling. The larger dining room is expected to generate net profits of $12,000 a year for the first 4 years and $15,000 a year for the next 6 years. At the end of 10 years, you will retire. You expect to be able to sell the restaurant for $70,000. You want to earn a minimum of 20% on your investment.

a. What is the net present value?
b. What is the Internal Rate of Return?
c. Should you expand the restaurant? Why or why not?
d. You hear news reports that real estate prices may rise over the next few years and that this is especially true for commercial real estate. You now think you can sell the restaurant for more than $70,000. What is the minimum selling price you must have so that you would be willing to expand the restaurant? Round to the nearest $.

[20] You have decided to start a business selling vitamin supplements. You estimate that you will have $200,000 in start-up costs. Annual expenses are expected to be $30,000 and revenue is expected to be $50,000 per year for the first 2 years and $75,000 in each of the next 3 years. You plan to sell the business at the end of *5* years for an estimated $250,000. You want to earn at least a 15% rate of return on your investment (MARR is 15%).

a. Should you invest? What is the IRR?
b. What is the NPV? Should you invest?
c. Your accountant tells you that your estimated selling price is too high. What is the minimum selling price you could accept and still have this investment be worthwhile?
d. One of the supplements you were going to sell has recently been banned. This will cause a substantial decrease in your revenue. What is the maximum annual decrease in revenue you could withstand and still have this investment be worthwhile?

[21] You are operations manager for Flatugas, a small natural gas producer in the Peace River Region of British Columbia. You are considering investing in a new technology that captures more of the gas from the wellhead. The technology will cost $1,000,000 and operating costs are estimated at $25,000 per year payable at the beginning of each year. The technology should result in $250,000 in extra natural gas revenue per year and have the added benefit of reducing greenhouse gas emissions. The technology is expected to last for *5* years when it could be sold back to the manufacturer for a guaranteed price of $249,908.70. The company has a 15% MARR and requires a payback period of 4 years or less.

a. What is the payback period of the investment? Should you invest?
b. Calculate the IRR for the investment and explain how you would use IRR to make a decision on whether to invest or not.
c. Calculate the NPV. Should you invest?
d. The director of finance reviews your analysis and tells you that a MARR of 13% is more appropriate for the project. He also informs you that the company should be able to sell clean air credits under the Paris agreement. What annual revenue would

be required from the credits to make this a worthwhile investment at a 13% MARR? (Assume 5 equal payments at the end of the year).

22 The Blue Sky Resort plans to install a new chair lift to serve a new ski area. Construction of the lift is estimated to require an immediate outlay of $190,000. In addition, the company must spend $30,000 today and $30,000 per year for the next 3 years to clear and groom the new area. The life of the lift is estimated to be 11 years with a salvage value of $80,000. Profits from the lift (excluding the cost of grooming and clearing) are expected to be $40,000 per year for the first five years, and $70,000 per year for the next six years. The company wants to earn a minimum of 14% effective on the investment. All revenue occurs at the end of the year and all expenses are paid at the beginning of the year.

a. Determine the net present value. Should they go ahead with the project? Why or why not?
b. Determine the internal rate of return. Should they go ahead with the project? Why or why not?
c. Determine the payback.
d. You hear on the news that we might be in a recession for the next few years. You now wonder if your salvage value estimate of $80,000 is too high. What is the lowest salvage value you can accept and still undertake the project? Round to the nearest $.

23 A project is to cost $60,000 immediately and to produce net cash flows of $20,000 at the end of the first year, $30,000 at the end of the second year, and $25,000 at the end of the third year. At the end of the third year the business will be sold for $5,000.

a. If the company requires 15% effective, will it be able to achieve its goal?
b. Find the value of the bonus paid at the start which would cause the project to earn exactly 20% effective.

24 Due to a restricted capital budget, a company can undertake only one of the following 3-year projects. Both require an initial investment of $650,000 and will have no significant salvage value at the end. Project X is anticipated to have annual profits of $400,000, $300,000, and $200,000 in successive years, whereas Project Y’s only profit, $1.05 million, comes at the end of year three.

a. Calculate the IRR of each project. On the basis of their IRRs, which project should be selected?
b. On the basis of NPV, which project should be selected if the firm wants to earn at

least 14% effective on their investment?

c. On the basis of NPV, which project should be selected if the firm wants to earn at least 11% effective on their investment?

Notes

1.
   a. Final Balance = -$1225, so this investment earns more than 10% effective
   b. Final Balance = $1300, so this investment earns less than 20% effective
   c. Final Balance = $0, so this investment earns exactly 15% effective
2.
   a. NPV = $1,012.40
   b. NPV = -$902.78
   c. NPV = $0
3. NPV = -$3,861.76. The deal does not earn 10% effective MARR.
4.
   a. NPV = -$198.90 at 15% effective. The company does not achieve its objective.
   b. The NPV of the cash flows using *i* = 20% effective is -$5,138.88. To earn 20%, the NPV must = $0. Therefore the size of the bonus must be $5,138.89 to have a net project cost $54,861.11 instead of $60,000.
5. NPV = $135,994.73
6.
   a. NPV = -$9,985.01 at 12% effective and IRR = 6.8305% for this three year project.
   b. To earn 15% effective, NPV must = $0. Without the subsidy, the NPV at 15% is -$15,101.18. This is the PV of the required subsidy at the end of the third year. Therefore, the subsidy = future value of this amount at 15% in three years or $22,967.01.
7. NPV = $664,371.70. This is the amount PA can afford to bid.
8. IRR = 12.7118%
9. NPV at 15% effective = $6,150.15. TQ will make more than 15% effective. IRR = 19.6004%
10. NPV = $9,076.27; IRR = 17.29597% effective.
11. NPV = $22,476.94 IRR = 10.5635%
12. NPVs are $100,000, $118,896.51 and $122,241.28 respectively. Offer #3 is the most attractive one at 9% effective.
13. A 10% bond implies that if the bond is purchased for $1,000, it will generate $100

annual interest, and the bondholder will receive the $1,000 bond investment back at the end of the ninth year. To solve this problem, find the PV of nine annual cash flows of $100 and the $1,000 paid at the end of the ninth year using the target interest rate. a. $761.42 b. $1,000.00 c. $1,355.39

14. NPV = $5,375.37 Yes, the company should purchase the equipment.
15. NPV for A is $39,162.96 NPV for B is $8,508.47. Location A has the higher NPV at 16% effective.

16.
   a. 4+136/297 years = 4.46 years
   b. -$17,152.42 ? No, because the NPV is negative.
   c. The NPV is +$1,111.42 which is above zero so you should open a kennel.
   d. $17,152.42
   e. minimum selling price is $255,000 + $33,025.52 = $288,026
   f. 12.12%, No because the IRR is below the minimum acceptable rate of return of 14%.

17.
   a. 4+30/170 = 4.176 years
   b. $16,263.94 Yes buy the business because the NPV is positive (earning more than 16% per year).
   c. Purchase price could increase by up to $16,263.94 to $136,264.
   d. Lowest selling price you could accept is $95,840.
   e. 19.45%/year. This exceeds the required rate of 16%/year so buy business.
   f. Up to a $4,967.16/year decrease in revenue.

18.
   a. NPV = +$6408 .24
   b. IRR = 16.73%
   c. Yes, undertake the business plan because the NPV is positive and the IRR is 16.73% which exceeds the minimum required return of 15%.
   d. $100,000-$14,822.65=$85,177 is the minimum selling price for the business.

19.
   a. -$18,573.73
   b. 14.77%
   c. No, because the NPV is negative and the IRR = 14.77% < MARR of 20%.
   d. $70,000 + $115,004 = $185,004 is the minimum selling price.

20.
   a. 17.25% > 15% MARR so yes invest.
   b. +$19,413.74. Yes, since the NPV is positive.
   c. $210,952 (rounded to nearest dollar)
   d. $5,791.42/year

21.
   a. 4.25 years so no, it is longer than 4 years.

b. 9.95% is below 15% so no, do not invest.
c. -$134,086.89 which is below zero, therefore do not invest
d. $24,000/year

22.

- NPV =+$7,979.31; yes, undertake the project because the NPV is positive
- IRR = 14.58% exceeds the MARR of 14%
- 6.57 years
- $80,000 -\33,722.42 =$46,277.58 is the minimum salvage value.

23.

- NPV = -$198.90; no, the company does not achieve its objective.
- The bonus must be $5,138.89 to make the NPV equal zero

24.

1. Project X: 20.8%   Project Y: 17.3% Project X has the higher IRR and should be selected.
2. NPV: Project X: $66,712   Project Y: $58,720 Project X has the higher NPV and should be selected.
3. NPV: Project X: $100,085   Project Y: $117,751 Project Y now has the higher NPV and should be selected.

Chapter 7: Extra Problems

The following groups of questions may be useful in studying for tests.

Chapter 1 and 2 Review

1 Gems Inc., an upscale jewelry store, purchased a diamond ring for $2,500 less 40% and 5%. The store's average per unit operating expenses (overhead) is 30% of cost. The "regular selling price" of the ring is established so that if the ring is sold in a "20% off sale" the net profit at the reduced price will be 20% of cost.

a. What is the reduced price of a ring in a "20% off' sale?
b. What is the "regular price" of the ring?
c. What is the net profit if the ring is sold at the "regular price"?

2 Samsong Inc., a TV **manufacturer** lists its deluxe models for $450 each, less trade discounts of 15% and 8%. A **retailer** wants to make a **net** profit of 10% of the selling price. If expenses are 15% of the selling price, at what price must he sell the TV?

3 The Alfacenturi Corp received an invoice dated July 12th for $5,400 with terms 3/10, 1.5/20, n/45. ABC made a payment of $2,000 on July 20th, and a second payment on July 31st that reduced the balance owing to $1,000. Find the size of the second payment.

4 An item that normally sells for $550 is put on sale for $357.50. Find the rate of markdown.

5 Radio Shock Electronics Co. makes televisions. Radio Shock sold a number of TV sets to a wholesaler at $399.90 per set, after discount rates of 15%, 9% and 6%.

a. What is the list price of a TV set?
b. What is the size of the 3rd discount (in dollars)?

6 A retailer has a policy of maintaining a margin of 60% on all items.

a. What is their rate of markup?
b. Another company maintains a 60% rate of markup on all items. What is their percent margin?

[7] A company purchases a line of basketball sneakers for $60 from Niko Inc. and sells them for $120.

a. What is the markup in dollars?
b. What is the percent margin?
c. What is the rate of markup?

[8] A pair of radically shaped skis costs the retailer $600 less chain discounts of 50%, 30% and 10%. The retailer maintained a 65% margin on all items. Since spring was fast approaching the retailer drastically reduced the selling price of the skis by 60%.

a. What was the regular selling price?
b. What is the sale price?
c. What is the rate of markup? (use the sale price not the regular price)

[9] What is the cost of an item that sells for $80 if the rate of markup is 60%?

[10] You know that a retailer makes $450 on the sale of a snowboard and the retailer has a rate of markup of 60%.

a. What is the selling price of a snowboard?
b. What is the percent margin?

[11] Filters-R-Us makes plastic coffee filters . It costs the company $6,500 to make 2,000 filters and $8,000 to make 4,000 filters. Assume the relationship between cost and the number of coffee filters produced is linear.

a. Find an equation that determines the cost, based on the number of filters produced.
b. How much would it cost to produce 3,000 coffee filters?

[12] Dick, Ed and Fran formed a partnership. The partnership agreement requires them to provide capital when and as required by the partnership in the ratio of 7:9:8, respectively.

a. If the total required initial investment was $96,000, how much did each contribute?
b. One year later, Dick's share of another injection of capital was $10,500.
    1. What is the total investment made by the partnership including the initial investment?
    2. What is Ed's share of the entire investment?

[13] At the Mac's store in Burnaby a 500 ml bottle of sparkling water sells for 1.19CAD. Convert this price to US dollars per gallon.

Rates:

- 1 CAD= 0.7125 USD
- 1.0567 Quarts= 1 Litre
- 1 Gallon = 4 Quarts
- 1,000 ml= 1 Litre

[14] A diamond ring cost a jeweler $4,200. He requires a margin of 45%.

a. At what price should he sell the ring?
b. What rate of markup did he realize?

[15] You are planning to open a neighbourhood ice cream shop and are doing some financial analysis. You can lease the shop for $4,300 for one month, salaries will cost $12,100 per month, hydro and miscellaneous expenses will be $450 per month. You can sell ice cream cones for $4.75 each. Each ice cream cone costs you $2.05.

a. Write down the revenue and cost equations.
b. Determine the number of ice cream cones you must sell in a month to break even. What are your total sales (in dollars) at the breakeven point?
c. You'd like to make a profit of $4,000 in one month. Determine the number of ice cream cones that you would need to sell.
d. Find the profit/loss if monthly sales (in dollars) are $14,250.

[16] Liquidation Electronics sells an article for $1,020.00 less 25% and 15%. A competitor

carries the same article for $927.00 less 25%. What further discount must the competitor allow so that its net price is the same as the discount store's?

[17] Find the equation of the line that passes through the points (-2, 5) and (3, 2).

[18] The license to operate a taxicab costs $2,000 per month. Insurance is $250 per month and maintenance averages $225 per month. Fuel costs $0.20 per kilometre. Taximeter rates are regulated and set at $0.45 per kilometre.

a. Determine the breakeven point for operating a taxicab.
b. What are the units for *x*, and what are the units for the slope?
c. What will be the profit/loss if 4,900 km are driven?
d. How many kilometres must be driven to make a profit of $2,500?

[19] Forever-Green Press prints its elegant advertising calendars in November and sells them for $9.60 each. The Company's production capacity for this run is 60,000 calendars at a variable cost of $3.80 each. The fixed costs (including artwork and licensing fees) allocated to this production run are $232,000. Assume that the company can sell all the calendars it produces.

a. What is the breakeven point in (a) units; (b) sales; dollars; (c) percent of capacity?
b. How many calendars must be printed and sold if they want to make a profit of $87,000?
c. If the plant operates at full capacity and November's entire calendar production run is sold, what profit (or loss) will be made?
d. Graph the cost and revenue equations (same graph). Mark the Breakeven Point on the graph.
e. If the variable cost increases by 25%, the fixed costs increase to $240,000, what new selling price should they charge if they want to earn a profit of $50,000, from the sale of 40,000 calendars?

[20] Plutonium fuel rods are sold at a list price of $5,000 with chained discounts of 15% and 25%.

a. Calculate the net price of a fuel rod.
b. Calculate the single equivalent discount rate.

c. The company selling the fuel rods would like to add a third discount to bring the overall discount rate to 40%. How large should the third discount be?

d. A competing firm offers a single discount of 35%. This amounts to a discount of $1,680 off the list price. What is the list price?

[21] Two long distance phone companies offer the following quotes:

- PLAN A: $22 per month plus $0.04 per minute.
- PLAN B: $15 per month plus $0.08 per minute.

Determine the point of indifference (i.e., the number of minutes per month where the costs are equal).

[22] Compute the single discount equivalent to the discount series 40%, 10%, 5%.

[23] An invoice for $3,200.00, dated April 20, has terms 2/10, n/30. What payment must be made on April 30th to reduce the debt to $1,200.00?

[24] On May 30th, Restorative Services Ltd. received an invoice for $2,600, terms 1.5/10, n/30. On June 9, Restorative Services made a partial payment of $1,379.00 on the invoice. How much is still owing after the payment?

[25] Albert, Bob and Chuck form a partnership and agree that half of the annual profit be distributed in proportion to the total number of hours worked in the business during the year and the other half in the ratio 4:3:2, respectively. The hours of work for Albert, Bob and Chuck were 300, 200 and 100, respectively. How much of the $180,000 profit should Chuck receive?

[26] The regular selling price of merchandise sold in a store includes a margin of 60%. During a sale, an item that cost the store $250 was marked down 40%. For how much was the item sold?

[27] An item that cost the dealer $500 less 20%, 15%, carries a price tag at a markup of 150% of cost. For quick sale, the item was reduced 30%. What was the sale price?

28 An appliance shop reduces the price of an appliance for quick sale from $1,560.00 to $1,195.00. Compute the rate of markdown.

29 Log-it-All Forest Products sells Grade-A sheets of plywood at trade discounts of 25%, 3%. A competitor has been selling the plywood at the same list prices but with trade discounts of 20%, 12.5%. Log-it-All wants to beat the competitor's prices by offering a third trade discount. At least how big must the additional discount rate be to meet this objective?

30 What amount must be remitted if the following invoices, all with terms 4/10, 2/30, n/60, is paid on May 10?

- $850.00 less 20%, 10% dated March 21
- $960.00 less 30%, 16.67% dated April 10
- $1040.00 less 33.33%, 25%, 5% dated April 30

Notes

1.
   a. $2,137.50
   b. $2,671.88
   c. $819.38
2. $469.20
3. $2,303.07
4. 35%
5.
   a. $550
   b. $25.53
6.
   a. 150%
   b. 37.5%
7.
   a. $60
   b. 50%
   c. 100%
8.
   a. $540
   b. $216
   c. 14.3%
9. $50

10. a. $1,200
    b. 37.5%
11. a. C = $5,000 + $0.75*x*, *x* = the number of filters produced.
    b. $7,250
12. a. Dick: $28,000, Ed: $36,000, Fran: $32,000
    b. $132,000 in total; Ed $49,500
13. 6.42 USD per Gallon
14. a. $7,636.36
    b. 81.82%
15. a. R = $4.75*x*, C = $16,850 + $2.05*x* where *x* = the number of ice cream cones.
    b. 6241 ice cream cones, $29,644.75
    c. 7,723 ice cream cones
    d. loss of $8,750
16. 6.47% or 6.5%
17. $y=3.8 - 0.6x$
18. a. 9,900 km
    b. *x* = number of km driven, slope= $/km
    c. -$1,250
    d. 19,900 km
19. a. 40,000 calendars; $384,000; 66.7% of capacity
    b. 55,000 calendars
    c. $116,000
    d. $12/calendar
20. a. $3,187.50
    b. 36.25%
    c. 5.8824% (or 5.88)
    d. $4,800
21. 175 minutes
22. 48.7%
23. $1,960
24. $1,200
25. $35,000
26. $375
27. $595
28. 23.4%

29. slightly more than 3.78%
30. $1,635.04

Chapter 3 and 4 Review

[1] Sam agrees to make a payment of $1,000.00 in 3 months at 8% simple interest. How much money is Sam borrowing today?

[2] Alixe borrowed $5,000 today at 12% simple interest. How much will she owe the bank when she pays off the debt in 180 days?

[3] Myrna borrowed $5,000 from the bank some time ago at 9% simple interest. She paid off the loan today with one payment of $5,675 . How many months ago did she borrow the $5,000? (Hint: it's easier to use $I = Prt$.)

[4] You would like to save to purchase a Kia automobile. You deposit $3,500 into a GIC that pays 8% interest, compounded quarterly. You need to save at least $5,000 to buy the car. If you make no more contributions, how many years will it take you to reach your goal?

[5] Lindsay made an investment of $1,500.00 made 42 months ago. It is now worth $2,110.65. What nominal rate of interest, compounded semi-annually, did this investment earn?

[6] Answer the following

a. What nominal rate, compounded semi-annually, is equivalent to 9% compounded quarterly?
b. What nominal rate, compounded monthly, is equivalent to 9% compounded quarterly?
c. What nominal rate, compounded quarterly, is equivalent to 12% compounded annually?

[7] A debt of $6,000 due today and $8,000 due 2.5 years from today is to be paid off with two payments. The first payment is to be made six months from today, and a second payment,

$5,000 larger than the first, 15 months from today. What should the two payments be if money is worth 9%, compounded quarterly? Use **15 months** as the focal date.

[8] Pavel was supposed to pay $10,000 today. Instead, he arranged with the bank to pay $3,000 12 months from today, followed by two equal payments in 18 months and in 30 months from today. The interest rate is 8% compounded quarterly. Calculate the size of each payment? Use **30 months** as the focal date.

[9] Anna borrowed $3,000 from her line of credit 8 months ago and a further $5,000 3 months ago . She arranged with the bank to pay off the line of credit with two payments. The first payment, 6 months from today, will be twice as large as the second payment made one year from today. The bank charges you 9% interest, compounded monthly. Use **6 months** as the focal date.

a. Find the size of each payment.
b. How much interest will she pay?

[10] A debt of $15,000 is due today. Instead, the borrower agrees to make one payment of $5,000 in 6 months and 2 equal payments in 3 months and 12 months from today, with simple interest charged at 6%. What is the size of each payment? Use **6 months** as the focal date.

[11] Matt is going to purchase a car and is given two options to pay.

- He can pay $25,000 in cash today, or
- $12,000 after one year, and a second payment of $15,000 after 18 months.

Which option is better? Your answer should be stated in terms of today’s dollars. The interest rate is 7% compounded semi annually?

[12] Mortel Inc. borrows some money today. In 10 months it must pay back $854,962.11. How much did Mortel borrow today if it is being charged 8% interest, compounded monthly?

[13] An investment of $5,000 made 27 months ago is now worth $7,756.64. What nominal rate of interest, compounded quarterly, did the investment earn?

14 Julia is seeking an investment and has narrowed her search to three different funds.

Fund A offers a return of 12.2% compounded quarterly.

- Fund B offers a rate of 12.4% compounded semi- annually.
- Fund Coffers a rate of 12% compounded monthly.

Convert all rates to effective rates and explain which rate is the best investment.

15. Serena invests $3,000 in a bond fund today. Her investment earns 4% compounded semi-annually in the first year, 6% compounded quarterly in the second year and 8% compounded annually in the third year.

a. How much would the $3,000 investment be worth at the end of the third year?
b. If, in the fourth year the bond fund loses money and the value of her investment falls by $200, what average annual rate of return, compounded quarterly, did she earn for the four years that she held the investment?
c. A friend has invested in a different bond fund and says she tripled her money in eight years. What nominal rate of interest, compounded semi-annually, did she earn?
d. At 9% compounded quarterly, how long would it take you to triple your money?

16 Mike purchased a two-year term deposit that pays interest at 5.7% compounded quarterly. At the end of the two years he renews the term deposit plus accumulated interest at 6.2% compounded semi-annually for an additional three years. At the end of the five years he has $6,724.84? How much did Mike initially pay for the term deposit?

17 Yolanda has a debt that comes due on November 26th and another that comes due on April 6th of the following year. She would like to celebrate Valentines Day by paying off the debts with a single payment on February 14th. How much should she pay if interest is 5% simple and each debt was for $1,000? (Use February 14th as the focal date.)

18 Davis borrowed $5,000, and 18 months later repaid the loan with a single payment of $5,600. Calculate the simple interest rate.

[19] Your investment of $8,500 grew to $8,893.09 at an interest rate of 4% simple. How many days was it invested?

[20] How much money do you need to deposit to earn $850 in interest in 18 months with interest at 6% compounded quarterly?

Notes

1. $980.39
2. $5,295.89
3. 18 months
4. 4.5 years
5. j_2=10.0000%
6.
   a. j_2 = 9.1012%
   b. j_{12}=8.9333%
   c. j_4=11.4949%
7. First payment: $4,284.02; second payment: $9,284.02
8. $4,231.33 is paid at 18 and 30 months
9. a. First payment: $5,871.56; second payment $2,935.78. b. $807.34
10. $5,262.17
11. In today ' s dollars, second option is worth $24,731.27< $25,000 , second option is cheaper.
12. $800,000
13. j_4 = 20.0000%
14. A: 12.7696%, B: 12.7844%, C: 12.6825% , B is the best investment.
15.
   a. $3,577.75
   b. j_4= 2.9759%
   c. j_2= 14.2151%
   d. 12.34 years
16. $5,000
17. $2,004.02
18. 8.0%
19. 422 days
20. $9,096.43

Chapter 5 Review

1 What will be the amount in an RRSP at the end of 20 years if monthly contributions of $500 are made at the end of each month and the RRSP earns 6% compounded monthly for the first 15 years and 7.5% compounded monthly for the remaining 5 years?

2 You have $4,000 in your savings account today. You will deposit $500 per month starting one month from now.

a. How much will you have in your savings account at the end of 5 years if you earn 6% compounded monthly?
b. How much interest would you earn?

3 You have just purchased a preferred share for $50.00. The company pays a dividend of $1.00 every quarter. One year later, when you sell the share, the interest rate is 10% compounded quarterly. How much will you gain or lose? (The next dividend is due in three months.)

4 You purchase a car and finance $10,000. The loan is to be repaid with monthly payments of $200 made at the end of the month for 5 years. What effective rate of interest is being charged?

5. You begin a savings plan. **Starting now** and for ten years, you deposit $100 per month into an account that pays = 6%. After 10 years, how much interest will you have earned?

6. Upon graduation, you have a student loan of $15,000. The most you can afford to pay is $550 per month. How long will it take you to repay the loan with payments of $550 per month starting in one month if the interest rate is 6% compounded monthly?

7 You purchase a car for $25,000 and make a 20% down payment. Interest is charged at 6% compounded monthly, and you will make 6 years of monthly payments. Find the size of the monthly payment if your first payment is in one month. (Round payment up to next cent.)

[8] You have decided to take a passive approach to fitness and order the ‘Ab-Cruncher’ from the shopping network. The Ab-Cruncher can be purchased for four easy monthly payments of $19.95 starting in one month. If interest is charged at 112 = 12%, find the cost of financing.

[9] You are 25-years-old and want to retire when you turn 55. Starting today you will deposit $200 into an RRSP every month for 30 years. You will then use the accumulated funds to purchase a 10-year annuity with the first withdrawal one month after your 55th birthday. Assume that the RRSP and the funds invested in the annuity earns 6% compounded monthly.

a. Find the size of the monthly withdrawals.
b. You decide that you will need $3,000 per month to live on when you retire at age 55. How much extra money must you contribute to your RRSP every month, so you can withdraw $3,000 every month for 10 years?

[10] You contribute $2,500 into your RRSP at the end of every quarter for 5 years. If your RRSP earns 8% compounded quarterly, how much interest will you earn in the 5 years?

[11] You borrow $50,000 and agree to make monthly payments for 15 years with the first payment one month from now. Calculate the size of your monthly payments if the interest rate is 9% effective.

[12] Paula, a BCIT business student, borrowed $5,000 from her parents to cover expenses and agreed to repay the debt with 20 equal monthly payments with the first payment due in 2 years. The interest rate is 12% compounded monthly.

a. How large is each payment?
b. How much interest did Paula pay her parents?

[13] Barton Fink bought a second hand car with $1,000 down and signed a contract agreeing to pay $299 every month for 3 years at 9% compounded monthly. The first payment is made one month later.

a. What is the cash price of the car?
b. How much interest will Barton have paid over the 3 years?

14 A debt of $5,000 is to be paid off by making payments of $502.31 at the end of every 3 months. If interest is 12% compounded quarterly, how long will it take to pay off this debt?

15 You need to set aside $2,500 for your vacation in Moose Jaw to celebrate graduation from BCIT 21 months from today. If your bank pays 9% compounded quarterly, how much must you deposit into your account at the end of every quarter to have $2,500 in 21 months?

16 A home entertainment centre may be purchased for $2,299 or by making payments of $199 to be made at the end of every month for 12 months.

a. What nominal rate of interest, compounded monthly, is being charged?
b. What is the effective rate of interest rate charged?

17 Fred Rock will deposit $500 in his RRSP every month for 15 years starting today. One month after the last deposit, he will withdraw the money in equal monthly withdrawal for 10 years.

a. Find the size of the monthly withdrawals if the invested funds earn 7.0% compounded monthly.
b. How much interest will Fred earn over the 25 years?

18 Sam Shepherd wants to borrow some money from the Federal Business Development Bank to open a doggy daycare business. The terms of the loan specify that no payments are required in the first year of the business. Starting in the second year, payments of $1,000 at the end of every three months are required for a period of five years (i.e., until the end of the sixth year of the business). If interest is 8% compounded quarterly, how much money is he borrowing?

19 Sam N Ella has just turned 50 years old today. He has decided to start saving for his retirement. He plans to retire when he turns 65. He will set aside $1,500 per quarter, starting today, into a money market account that earns 4% compounded monthly.

a. How much will Sam have on the day he retires?

b. How much interest will he have earned?

[20] Judith Fortune recently won $500,000 from a hospital lottery. She has decided to put some of her winnings into an RRSP. Miss Fortune is 55 years old and plans to retire at age 65. She would like to receive $2,000 per month for twenty years starting on her 65th birthday.

a. If she can earn 6%, compounded monthly, how much of her winnings should she put into her RRSP today?
b. Miss Fortune has changed her mind and has decided to put half of her winnings into an RRSP instead. How much extra will she receive per month when she retires?

[21] A dining room suite sells for $4,999. It may be purchased from Junk-O Furniture by making a down payment, followed by monthly payments of $245 for a year-and-half. (The first payment is one month later.) Interest is j_{12} = 12%. Find the size of the down payment.

[22] An alumnus wants to donate a sum of money to his Alma Mater that will provide a scholarship of $750.00 every 6 months in perpetuity. If money can be invested at 6% compounded semi-annually and the first $750.00 is to be awarded at the end of six months how much must he donate to the school today?

[23] You have decided to purchase preferred shares of No-Vision Eye Treatment Centre that pays a semi-annual dividend of $1.75 per share. Current interest rates are 7% compounded semi-annually and the next dividend is due in six months.

a. What should you be willing to pay per share?
b. Two years later the interest rates drop to 5% compounded semi-annually. You have decided to sell your shares. Calculate your gain or loss per share if the next dividend is due in six months – and the dividend per share remains the same.

[24] You purchase 250 preferred shares of Silent Witness Security Systems Inc. that pays a dividend of $2.25 per share every 3 months. Current interest rates j_4 = 6%.

a. What would you be willing to pay for the 250 shares?
b. If interest rates rise to 8% compounded quarterly, how much would you expect to gain or lose if you sell all of your shares? **Note:** the next dividend is due in 3 months and the dividend per share is still $2.25.

Notes

1. $247,586.15
2.
   a. a. $40,280.42
   b. b. $6,280.42
3. Loss of $10
4. $j_1 = 7.6777\%$
5. $4,469.87
6. 30 months
7. $331.46
8. $1.96
9.
   a. $2,241.59 per month
   b. $267.67−$200=$67.67 /month
10. $10,743.42
11. $496.74
12.
   a. $348.33
   b. $1,966.60
13.
   a. $10,402.59
   b. $1,361.41
14. 3 years
15. $337.75
16. .
   a. 7.0708%
   b. 7.3045%
17.
   a. $1,840.10
   b. $1,840.10× 120- $500×180 = $130,812
18. $15,106.20
19.
   a. $123,866.46
   b. $33,866.46
20. .
   a. a. $154,203.50
   b. b. $3,242.47 - 2,000 = $1,242.47
21. $981.42
22. $25,000

23. .
    a. $50/share
    b. Gain of $20/share
24. .
    a. $37,500
    b. $9,375 (loss)

Entire Course Review

[1] The executor of Sam Jackson's estate is to divide $880,000 between three charities, the United Way, Heart Fund and Red Cross in the ratio of 6:3:2 respectively. How many dollars will each charity receive?

[2] A publisher sells its romance novels with chained discounts of 10% and 20%.

a. Find the single equivalent discount rate.
b. The publisher would like to add a third chained discount to bring the overall discount to 31.6%. Find the third chained discount rate.

[3] An invoice with terms 2/10, 1/20, n/45 for $3,000 dated November l was partially paid by a $1,470 payment on November 10th and a second payment of $495 on November 20th. What is the outstanding balance after the November 20th payment?

[4] Black Bear Cupcakes received an invoice dated November 28th with terms 1.5/10, n/30 for $8,050. On December 6th, Black Bear mailed a cheque for $4,925 in partial payment of the invoice. What is the outstanding balance after the December 6th payment?

[5] Chemco Co. sells its products at trade discounts of 25%, 10%. A competitor has been offering products at the same list prices but with trade discounts of 30%, 10%. Chemco wants to beat the competitor's prices by offering a third trade discount. At least how big must the additional discount rate be to meet this objective?

[6] What single discount is equivalent to a chain discount of 15%, 10%?

[7] If we want a single equivalent discount of 19.25% by using chain discounts of 15% and x, what must the second discount rate be?

[8] A furniture manufacturer offers his clients discount rates of 22.5% and 10%.

a. What single rate is equivalent to this series of rates?
b. If a client of this manufacturer paid $1,395 for a sofa, at what price was the sofa listed?

[9] Find the selling price of an item bought for $1,050.00 if the margin is 30%?

[10] If a company maintains a margin of 20% on all items it sells, what is the rate of markup?

[11] The regular selling price of an item sold in a store includes a margin of 60%. During a sale, an item that cost the store $240 was marked down 20%. For how much was the item sold?

[12]. The net price of an article is $612 after discounts of 15% and 10%. What was the list price?

[13] An item that cost the dealer $850 less 35%, 20% carries a price tag at a markup of 25% of cost. For quick sale, the item was reduced 25%. What was the sale price?

[14] Find the cost of an item sold for $2,800 to realize a rate of markup of 40%.

[15] An article cost $300 and sold for $450. What was the rate of markup?

[16] The markup on an item is $50. If the margin was 40%, what was the cost?

[17] The price of an item is reduced for quick sale from $950 to $760. Compute the rate of markdown .

[18] You are taking a holiday in Britain. You are taking 850 US dollars with you. How many British pounds can you buy with this money?

Rates:

- 1 USD= 1.38358 CAD
- 1 British pound(£) =$2.21824 CAD

[19]. A manufacturer of major appliances provides the following information about the operations of the refrigeration division:

- Fixed costs per period are $26,880
- Variable costs per unit are $360
- Selling price per unit is $640, and
- Capacity is 150 units.

Compute the break-even point:

a. in units.
b. as a percent of capacity.
c. in dollars.

[20] Find the cost-equation for shipping one ton of a product if we know that it costs $500 to ship it a distance of1,000 kilometres and $750 to ship it 2,000 kilometres. Assume that the relationship between cost of shipping and distance shipped is linear. What would it cost us to ship one ton of the product 1,500 kilometres? Check your answer by putting the two points back into the equation.

[21] The High Tech computer shop assembles and sells computers. After reviewing their accounting data, the following was determined.

- Fixed costs: $100,000 per year
- Variable costs $1,450 per computer
- Selling price per computer: $2,450
- Capacity is 1,000 computers per year

a. How many computers must it sell per year to break even?

b. What is the total yearly sales (revenue) required to break even?

c. Determine the computer shop's yearly profit if it sells 250 computers per year?

d. How many computers would need to be sold to make a $25,000 profit?

e. If the variable costs increase to $1,600 per computer, what should you increase the selling price to if you want to earn a profit of $25,000 from the sale of 125 computers?

[22] A small production company has 3 choices when renting a mobile crane for short periods:

- Choice A:It can pay $200 per hour used.
- Choice B: Pay an annual fee of $5,000 and pay only $100 per hour used.
- Choice C: Pay an annual fee of $12,000 and pay only $50 per hour used.

a. Write the cost functions for each alternative.

b. Calculate the points of indifference (where the costs are equal).

    i. between alternative A and B

    ii. between alternative B and C

c. Determine the range of hourly usage where the three alternatives would be preferred.

[23] You take out a loan for $10,000. The most you can afford to pay is $425 per month at the end of every month. If the interest rate on the loan is 7% compounded monthly, how many months will it take you to pay off the loan? (No decimal answers!)

[24] Kate made $600 monthly deposits at the end of every month for 5 years into an RRSP paying 8% compounded monthly. Immediately after the end of the 5th year the rate is reduced to 6% compounded quarterly.

a. If neither deposits nor withdrawals were made during the next 10 years, how much would Kate have in her account at the end of 15 years?

b. How much interest did Kate earn over the entire 15 year period of time?

[25] You have just turned 30 years old. Starting today, you will make monthly contributions of

$600 into a RRSP for 25 years. One month after your 55th birthday you will begin taking monthly withdrawals with your last withdrawal on your 70th birthday.

a. Find the size of these withdrawals if the interest rate is 6.5% compounded monthly.
b. How much interest did you earn over the entire 45 years?

[26] Barney wants to start saving for his retirement. Starting today, he will make monthly deposits to his RRSP for 20 years. One month after his last deposit, he wants to withdraw $4,000 per month for 10 years. Assume the invested funds earn 6% compounded monthly for the entire time.

a. How much must Barney deposit into his RRSP per month to achieve his goal?
b. How much interest will Barney earn over the 30 years?

[27] If you agree to make a payment of $1,500 in 3 months at 6% simple interest, how much money are you borrowing today?

[28] You borrow $6,000 on November 15, 2021 at 8% simple interest. How much do you owe the bank when you pay off the debt on March 15, 2022? What is the cost of financing?

[29] You borrowed $3,000 from the bank some time ago at 6% simple interest. You paid off the loan today with one payment of $3,360. How many months ago did you borrow the $3,000?

[30] A debt of $6,000 was due 6 months ago and a debt of $14,000 is due 5 months from today. Instead, the borrower agrees to make 2 equal payments, to be made 3 months, and 8 months from today, with simple interest allowed at 12%. What is the size of the payments? Use 5 months as the focal date.

[31] You would like to save to return to school. You deposit $4,000 into a GIC that pays j_4 = 7.44%. You have decided to return to school when your savings grow to at least $6,000. If you make no more contributions, how many **years** will it take you to reach your goal?

32 An investment of $2,000 made 30 months ago is now worth $2,676.45. What nominal rate of interest, compounded semi annually, did the investment earn?

33 What nominal rate, compounded semi-annually, is equivalent to 12% compounded quarterly?

34 Debts of $800 due today and $900 due in 27 months are to be repaid with 2 equal payments, 1 year and 2 years from today. If the interest rate is 9% compounded quarterly, find the size of the payments. Use 2 years as the focal date.

35 You are searching for a mutual fund.

- Fund 1 offers a rate of return of 6.4% compounded quarterly.
- Fund 2 offers a rate of return of 6.6% compounded semi-annually.

Convert all rates to effective rates. Which fund is the better investment?

36 You purchase a new car. The dealer offers you terms of 20% down and the remainder financed over three years at an interest rate of 9% **compounded quarterly.** The cost of the car is $21,640.79.

a. Find the size of your **monthly** payment if your first payment is due one month after you purchase the car.
b. What is the cost of financing (i.e., how much interest will you pay)?

37 A TV set may be purchased on these terms: a down payment of 20% of the selling price is to be made on the date of purchase, followed by 18 monthly payments of $40 each, with the first payment one month after the date of purchase. If the interest rate charged is 10% compounded monthly, what is the cash price (selling price) of the TV set?

38 A used car that sells for $15,000 may be purchased by making payments of $500 per month for 3 years with the first payment due the day the car is purchased.

a. What nominal rate of interest, compounded quarterly, is charged?

b. What effective rate of interest is charged?

[39] You take out a loan for $12,000 with quarterly payments of $806.59 at the end of each quarter. If the interest rate on the loan is 12% compounded quarterly, how many years will it take you to pay off the loan?

[40] John will purchase an annuity that will pay him $5,000 per quarter for 10 years beginning when he turns 60 years of age. If John's current age is 45 years and the invested funds will earn 7.0% compounded quarterly, what amount must he invest in the annuity today so he can collect $5,000 per quarter for 10 years with the first payment on his 60th birthday?

[41] A bursary fund for BCIT honour students is to be funded by a perpetual fund. The fund earns interest at 8% compounded semi-annually and is to pay $2,000 every six months, with the first scholarship paid in six months. Find the size of the initial funding that is required.

[42] A friend has invested in a bond fund and says she doubled her money in five years. What rate of interest, compounded semi annually, did she earn?

[43] A person can buy a piece of land for $130,000 now **or** $60,000 now and $100,000 in 5 years. Which option is better if money can be invested at:

a. 6% compounded quarterly?
b. 10% compounded quarterly?

[44] Meryl just turned 50 years old. She put $50,002 into her RRSP today. She will leave the money in her RRSP until her 65th birthday. She will then purchase a 10-year annuity with the first withdrawal one month later. Use

a. What will be the size of the monthly withdrawals?
b. How much interest did Meryl earn over the 25 years?
c. Meryl has determined that she requires a larger monthly payment than that found in part (a). She would like to receive $2,000 per month. How much additional money must she transfer to her RRSP today so she can collect $2,000 per month instead?

[45] You invest \$4,000 in a mutual fund. Your investment of \$4,000 earns the following returns.

Year	**Return**
Year 1	j_1 = 9%
Year 2	j_2=12%
Year 3	j_4 = 10%

What average nominal rate of return, compounded semi annually did you earn?

[46] You purchase a new car. The dealer requires that you put \$6,000 down followed by monthly payments of \$999 over four years. (The first payment is made one month after you buy the car). The interest rate is 9.9% effective.

a. What is the cash price (selling price) of the car?
b. What is the cost of financing?

[47] You contribute \$2,500 into your RRSP at the end of every quarter for 5 years. If your RRSP earns 8% compounded quarterly, how much interest will you earn in the 5 years?

[48] A computer that sells for \$3,999 may be purchased by making a down payment , plus a series of month-end payments of \$225 for one and a half years. If the interest rate is 9% compounded monthly, what is the size of the down payment?

[49] You are buying a house for \$260,000 with a down payment of 20%. The interest rate is 8.5% compounded semi-annually. The mortgage is amortized over 25 years for a 3-year term.

a. Calculate the size of the monthly payment. The lender's policy is to round payments up to the next whole dollar.
b. How many payments would be required?
c. How much interest will you pay in the first 3 years?
d. How much interest would you pay in the third year only?
e. You make an extra lump-sum payment of \$40,000 at the end of 3 years. What is the outstanding balance after this lump-sum payment?

f. When you go to renew your mortgage at the end of

g. 3 years, the rates have fallen to only 5.5% compounded semi-annually for a 3-year term. Find the size of the new payment.

h. The payment determined in part (f) is much smaller than you thought due to the lower rate and lump sum payment. You have decided to increase the monthly payment so that you pay off the remaining balance in only 12 years instead of 22 years. Find the size of the new payment. Round up to the next dollar.

i. Find the size of the final payment.

[50] A summer cottage, valued at $120,000, may be purchased by paying a $20,000 down payment and financing the balance with a mortgage at 9% compounded semi-annually and monthly payments for 15 years.

a. Find the monthly payment. Round up to the next dollar .

b. How much of the 60th payment pays interest and how much goes toward the principal?

c. After the 100th payment, how much of the original mortgage is still left to be paid?

d. After making 115 payments, what percent of the original debt will have been paid off?

[51] You are contemplating purchasing a business selling computer software over the Internet.

- You estimate that the purchase price would be $40,000.
- Your expenses should be $10,000 per year.
- You expect annual revenues to be $15,000 for the first two years and $20,000 each year after that.
- You plan to sell the business at the end of six years and estimate you will get $55,000.
- Your **MARR** is 15% effective.
- Assume all expenses are paid at the beginning of the year and revenues are received at the end of the year. **Time diagram is required.**

a. What is the IRR? Should you purchase the business? Why or why not?

b. Calculate the NPV. Should you purchase the business? Why or why not?

c. What is the highest purchase price you could pay and still be willing to buy the business?

d. What is the lowest selling price you could tolerate and still be willing to undertake the business? Round to the nearest dollar.
e. You accountant advises you that your annual revenue projections are too high. What is the maximum annual decrease in revenue you could withstand and still have this investment be worthwhile?

[52] A food concession at an airport has a 7 year life and costs $200,000. Renovations will cost you another $50,000. The concession's operation is expected to produce net incomes of $60,000 a year. The salvage value of the equipment and ending inventory are expected to total $40,000. Your **MARR** is 20%.

a. Find the IRR. Would you buy this concession? Why or why not?
b. Find the NPV. Would you buy this concession? Why or why not?
c. By how much does the purchase price of the concession have to fall to make the investment worthwhile? What is the new price?
d. Your accountant tells you your salvage estimate is too low. What is the minimum salvage value you require to make the investment worthwhile?
e. The Canadian government has decided to offer a one-time subsidy to encourage the creation of concession stands at the airport. The subsidy is received one year later. How large of a subsidy would you require to make the investment worthwhile?
f. Your accountant advises you that your projected net incomes are too low. What is the minimum annual increase in revenue you require to make the investment worthwhile?

Notes

1. United: $480,000; Heart: $240,000; Red Cross: $160,000.
2. a. 28% b. 5%
3. $1,000
4. $3,050
5. 6.67%
6. 23.5%
7. 5%
8. .
    a. 30.25%
    b. $2,000
9. $1,500
10. 25%

11. $480
12. $800
13. $414.38 is the sale price, the price before the 25% markdown was $552.50.
14. $2,000
15. 50%
16. $75
17. 20%
18. £530.17 (GBP)
19.
    a. in units, 96
    b. as a percent of capacity, 64%
    c. in dollars, $61,440.
20. C = $250 + $0.25*x* where *x* is the number of kilometers driven, $625.
21.
    a. 100 computers
    b. $245,000
    c. $150,000
    d. 125 computers
    e. $2,600
22.
    a. C_A=$200*x*; C_B =$5,000 +$100*x*; C_C=$12,000 +$50*x*, x = the number of hours
    b. i. 50 hours; ii. 140 hours
    c. < 50 hours use A; 50-140 hours use B; > 140 hours use C.
23. 26 months (The final payment will be smaller.)
24.
    a. $79,973.02
    b. $43,973.02
25. .
    a. $3,935.10
    b. $3,935.10× 180 - $600× 300 = $528,318
26. .
    a. $779.79/month
    b. $4,000×120 − $779.79× 240 = $292,850
27. $1,477.83
28. $6,157.81 and $157.81 is the interest
29. 24 months
30. $10,377.35
31. 5.5 years
32. 12.0%

33. 12.18%
34. $877.20
35. 6.5552%, 6.7089%, Fund two is highest.
36.
    a. $550;
    b. $2,487.37
37. $832.54
38.
    a. a. i/month = 1.083423742%, j_4 = 13.1425%
    b. j_1 = 13.8045%
39. 5 years
40. $51,370.97 today
41. $50,000
42. 14.35%
43.
    a. Financing more costly because $134,247 > $130,000
    b. Financing cheaper because $121,027 < $130,000.
44.
    a. $1,654
    b. $148,478
    c. $10,459.89
45. j_2=10.40678%
46.
    a. $45,781.31;
    b. $8,170.69
47. $10,743.42
48. $223.68
49.
    a. 1654.356115 → $1,655/month
    b. 300 payments (299 full-sized payments and 1 smaller final payment. (n = 299.6072073)
    c. $51,140.25
    d. $16,809.59
    e. $159,560.25
    f. 1037.575056 → $1,038/month
    g. $1511.061987 → $1,512/month
    h. $1,322.06
50. .
    a. $1,004.52 → $1,005/month
    b. $414.21 principal repaid, $590.79 interest
    c. $60,495.03
    d. 48.345%

51.
a. 18.62% > MARR of 15% so yes, buy the business.
b. $7817.58 > 0 so yes, buy the business.
c. up to $47,817.58
d. $55,000- $18,083 = $36,917
e. decrease of $2,065.69/year

52.
a. IRR= 16.83% < 20% MARR so no
b. No, NPV=-$22,561.23, NEG.
c. falls by $22,561 to $177,439
d. $120,841 ($80,841 increase)
e. $27,073.47
f. $6,259/year

Appendices

These are optional topics, which are not always covered.

Appendix A: Learning Curves in the BAII Plus

The BAII Plus calculator can be used for more than just Business Math. For example, we use power functions to understand learning Curves.

Example A.1

In manufacturing a certain part, we find that the initial part takes 176 hours to manufacture, and that there is a 90% learning curve.

a. Find a power function that shows the time to create the *n*th item as a function of *n*.
b. How much time does it take to manufacture the 144th item?
c. How many items to we need to manufacture before we are down to 50 hours per part?

We are looking for a function of the form:

$$t_n = t_1 n^b$$

or using your calculator's built in notation $y = ax^b$

We can find this using the Data worksheet on the calculator, by fitting a power function using the data points given: (1, 176) and (2, 176×0.9)

Step	To	Press	Display
1	Select Data Worksheet	[2ND] [7/DATA]	X01=0
2	Clear old Data	[2ND] [CE|C]	X01=0
3	Enter two data points	[1] [ENTER]	X01=1
↓176 [ENTER]	Y01=176		
↓2 [ENTER]	X02=2		
↓176[×].9 [=][ENTER]	Y02= 158.4		
5	Enter STAT mode	[2ND] [8/DATA]	LIN
6	Get into PWR mode	[2ND] [ENTER] [2ND][ENTER] [2ND[ENTER]	PWR
7	Check that you entered data correctly	[↓]	n=2
8	Find the Power Function	↓↓↓↓↓↓↓	a= 176
↓	b= -0.152003093		
9	Find the time for item 144	↓↓ 144 [ENTER]	X' = 144
↓ [CPT]	Y' = 82.68655388		
10	Find the number of the item that takes 50 hours	50 [ENTER]	Y'= 50
↑ [CPT]	X' = 3940.961945		

a. We have found a and b in the calculator, so we know that we have: $t_n = 176n^{-0.152003093}$
b. The 144th item takes 82.69 hours to manufacture
c. It will take around 50 hours to manufacture the 3941st item.

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
- These notes are for you only (they will not be stored anywhere)
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An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://pressbooks.bccampus.ca/businessmathematics/?p=1180#h5p-1

Appendix B: EUAC and Capitalized Cost

COST COMPARISONS

Some investment decisions involve the purchase of equipment which provides benefits that clearly outweigh the costs. For example, a sprinkler system may be required before a building can be used, a salesman may need an automobile, an office may need a copier. In such cases the issue is really that of obtaining the equipment at the lowest cost. Consequently, you need only examine those areas in which alternatives differ.

Example B.1

As an example, suppose that an office needs a heavy duty copier and has to choose between two alternatives, Model I and Model II. Both models will do a satisfactory job. The characteristics and costs of the two models follow:

	Model I	**Model II**
Useful Life	3 years	3 years
Price	\$8,500	\$10,000
Operating Cost	\$2,000/yr	\$1,500/yr
Disposal Value	\$1,500	\$2,000

Assume the operating cost is paid at the beginning of each year.

As with our earlier work, we must allow for the time value of money – amounts paid early are more expensive than amounts paid later since the funds to make later payments are assumed to be invested until they are needed. We will assume the company expects a rate of return of 12% effective.

All amounts should be discounted to find the Net Present Value of the cash flows.

For Model I, we have:

TIME	Cash Flow	PV
0	-$10,500	-$10,500.00
1	-2,000	– 1,785.71
2	-2,000	-1,594.39
3	1,500	1,067.67
	NPV =	-$12,812.43

For Model II, we have:

TIME	Cash Flow		PV
0	-$12,000		-$12,000.00
1	-1,500		-1,339.29
2	-1,500		-1,195.79
3	2,000		1,423.56
	NPV	=	-$13,111.52

These net present values represent the amount of money the company would have to set aside as a lump sum, which, with its earnings, would provide the payments necessary for the copiers. Clearly in this case Model I would be the cheapest.

If, however, no allowance was made for the time of payments and the amounts were simply totaled (equivalent to a zero rate of return), then the costs of the copiers would appear *exactly the same.*

EFFECTS OF DIFFERENT LIFETIMES

In the copier example above, the two copiers were expected to last for the same time. If alternatives are expected to last for different times, their costs can be compared by putting them on a per year or per month basis. To allow for the time value of money in such cases, you can use the annuity payments which would have as a present value the NPV of the costs. If this is done on an annual basis, the annuity payment is called the ***Equivalent Uniform Annual Cost (EUAC)***.

Example B.2

Suppose that there had been a third alternative copier with the following characteristics:

MODEL III

	Model I	**Model II**	**Model III**
Useful Life	3 years	3 years	4 years
Price	$8,500	$10,000	$11,000
Operating Cost	$2,000/yr	$1,500/yr	$2,000/ yr
Disposal Value	$1,500	$2,000	$1,500

This one would have to be compared with Model I (the cheaper of the other two) and the question would be: Is the extra initial cost of Model III justified by the longer useful life?

To answer this, start by finding the NPV of Model III costs.

Time (year)	**Cash Flow**	**PV**
0	-$13,000	-$13,000.00
1	-2,000	-1,785.71
2	-2,000	-1,594.39
3	-2,000	-1,423.56
4	1,500	953.28
	NPV	-$16,850.38

The NPV cannot be simply compared to that of Model I since the NPV is for a different time of use. Instead, we place it on an annual basis by finding the equivalent yearly annuity for each model.

For Model I:

$$n = 3,\ i = 12$$

from which you will find the payment is -$5,334.44.

This is the **Equivalent Uniform Annual Cost (EUAC),** i.e., the annual payment equivalent in value to the set of payments required for the given asset (in this case Model I).

For Model III:

$$n = 4,\ i = 12$$

from which the payment= EUAC = – $5,547.72.

Hence, this alternative would be more expensive than Model I.

CAPITALIZED COST

Another approach to dealing with unequal lives is to view the equipment as being replaced at the end of each cycle with an identical piece of equipment with identical costs. Then the NPV of all of the cash flows is found. Such an NPV is called the *Capitalized Cost.*

For Model I above this can be found by using the EUAC and the perpetuity formula:

$$PV = \frac{PMT}{i} = \frac{-\$5,334.44}{0.12} = -\$44,453.67$$

Similarly for Model III:

$$PV = \frac{PMT}{i} = \frac{-\$5,547.72}{0.12} = -\$46,231.00$$

again showing that Model I is cheapest.

The capitalized cost method of comparison is particularly suited to constructions such as bridges which are often viewed as never to be replaced, simply to be built and maintained.

Example B.3

Suppose a small bridge is to be built for $4,000,000 and to be maintained indefinitely for payments of $35,000 a year, starting at the end of the first year. Then the capitalized cost at a rate of return of 10% effective would be:

$$\text{Capitalized Cost} = \$4,000,000 + \frac{\$35,000}{0.10} = \$7,500,000$$

this being the amount of money set aside now which, with earnings, would make all payments for the bridge.

Your Own Notes

- Are there any notes you want to take from this section? Is there anything you'd like to copy and paste below?
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Glossary

—

Double dash (double en dash) is used in this textbook to denote 'N/A' or nothing needs to be entered.

2ND

The [2ND] function key (near the top left on the BAII Plus) allows the user to access the secondary menus. Example: [2ND][PMT] accesses the BGN menu.

accelerated bi-weekly

Take the monthly mortgage payment (P/Y=12), and divide by 2, than make this payment bi-weekly. (P/Y=26). This amounts to one extra monthly payment per year, which speeds up your mortgage repayment.

Acid-test ratio

Ratio of cash resources to the company's short-term debts.

amortization

The process of paying off debt through regular principal and interest payments over time.

amortization period

The total length of time until a loan is fully repaid.

amortization table

A table (or schedule) detailing the amount of principal and interest paid during each payment as well as the balance owing after each payment.

AMRT

Amortization menu in the BAII Plus. Hit 2ND PMT to access this menu.

annuity

A series of equal-sized payments, at regular intervals, over a fixed period of time.

annuity due

An annuity with deposits or payments at the beginning of the payment period.

BGN

The setting in the BAII Plus to turn on when payments occur at the beginning of the payment interval.

bond holder

Canada Education Savings Grant (CESG)

The CESG provides 20% of the Registered Education Savings Plan (RESP) contributions of up to $2,500. That means the CESG can add a maximum of $500 to an RESP each year.

Compound interest

interest paid on previously earned interest as well as on the original principal.

contribution margin

Markup in dollars for one item

cost of financing

The amount of interest we must repay on the loan.

Cost of the Goods Sold (COGS)

Total Variable expenses, or all the costs that go into making your item.

coupon payment

The regular (usually semi-annual) payment from a coupon bond.

coupon rate

In a bond, this is the initial interest rate used to calculate the coupon payment

CPT

Compute key on BAII Plus. Computes selected value.

defaults

To default on a loan is to fail to make the payments. This can lead to fines, legal procedures, or items being repossessed.

deferred annuities

An annuity where the regular payments are delayed for a period of time.

Dividend

The regular payment (PMT) paid by the issuer of the preferred share.

down payment

A lump-sum payment made before you take out a loan.

effective rate

The equivalent rate compunded annually

END

The setting in the BAII Plus to make the payments occur at the end of the payment interval.

ENTER

The [ENTER] key (top left on the BAII Plus) is used to set values within the calculator's menus.

Equivalent

This is a mathematical term, meaning that two things are the same in the ways we want them to. In the case of interest rates, two rates are equivalent if an investment at each rate gives the same Future Value after one year.

Equivalent Uniform Annual Cost (EUAC)

The Payment of an Ordinary Annuity with PV = NPV

Expense

A cost, either fixed or variable, Money Out

face value

In a bond, this is the original purchase price, as well as final payment for the bond.

fair market value

How much an item would sell for on the open market - normally, this is the PV according to today's interest rates for similar investments.

fair market value of a bond

The amount the purchaser (secondary bond holder) is willing to pay for a bond.

fixed cost

Also can be known as the Operations Costs or Operating Expenses. Stay the same no matter how may units are sold.

fixed interest rate

An interest rate which remains constant through the entire term, instead of fluctuating based on market conditions.

fixed rate mortgage

A mortgage where the interest rate charged remains fixed for the duration of the mortgage term.

FOCAL DATE

The date at which we make the calcualtions.... (FIX)

general annuity

An Annuity where $P/Y \neq C/Y$

Gross Profit (GP)

Revenue minus Cost of Goods (or Variable expenses) only. The Operating expenses or Fixed Costs are not considered.

interest

Money earned on an investment, or paid on a loan.

internal rate of return (IRR)

The rate for which the $NPV = 0$

issuer

lease payments

Equal sized payments for a lease.

lease period

Duration of a lease (fixed time period).

lessee

Person or company holding the lease. Example: person who is leasing a car.

loan

An annuity where we borrow an initial amount of money (PV) and we repay the loan with a series of equal-sized payments (PMT), at regular intervals, over the course of a fixed time period

Market Rate

The current interest rate.

markup

maturity date

The termination or ending date for which a loan, bond, or any amount borrowed must be paid back in full.

method of substitution

A method of solving a system of equations.

minimum acceptable rate of return (MARR)

mortgage term

The length of time your mortgage agreement and interest rate will be in effect.

Net Present Value (NPV)

The sum of all Inflows, minus all Outflows, adjusted for time.

Nominal rate

Percent annual rate in compound interest.

Operating expenses

Fixed Costs

ordinary annuity

An annuity where the payments occur at the end of each payment interval.

ordinary perpetuity

A perpetuity where the first payment comes at the *end* of the first period

partial payment

percent gross margin

Percent Markup

The ratio of Profit over Cost, (usually Gross Profit).

percent net margin

Periodic rate

Periodic Compound interest rate

perpetuity

An annuity that has no end or an annuity with regular cash flows that continue forever.

perpetuity due

A perpetuity where the first payment is at the *beginning* of the first period.

PMT

The 'payment' key within the TVM (time value of money keys) in the BAII Plus Calculator.

preferred share

A preferred share has no maturity date (no expiration date or fixed term) but will stop regular payments if the company stops making a profit of goes out of business. Because we have no way of knowing when this might happen, we treat preferred shares like perpetuities and assume the profits will continue infinitely (forever).

Price

The selling price, fair market value or the amount the purchaser is willing to pay.

principal

The original amount of money invested or borrowed.

Profit

The difference between the revenues (sales) and expenses (cost of goods and operating costs).

renew

A new mortgage is drawn up at the end of each term when a mortgage holder (buyer) renews their mortgage.

residual value

Size of final lease payment required. Example: final payment required to own a car at the end of the car lease.

RESP

Registered Education Savings Plan

return on investment (ROI)

revenue function

RRSP

Registered Retirement Savings Plan

secondary bond holder

A bond holder who has purchased a bond(s) from either the original bond holder or another secondary bond holder.

Sells at a Discount

Sells for a price lower than the original purchase price.

Sells at a Premium

Sells for a higher selling price than the original purchase price.

Semi-annual payments

Payments that occur twice per year or once every six months.

simple annuity

An annuity where the payment frequency (P/Y) is equal to the compounding frequency (C/Y).

simple interest

Interest earned without any compounding, that is interest paid only on the principal.

system of equations

A number of equations, involving the same variables

term of a bond

The amount of time between when the bond is issued and when the bond matures.

Turnover Ratio

The value of goods sold divided by the average value of goods in stock

TVM keys

Time Value of Money keys. These are the N, I/Y, PV, PMT and FV keys on the BAII Plus.

variable cost

Costs that vary based on how many items are made and sold.

variable rate

variable rate mortgage

the interest rate varies throughout the mortgage term

Formulas

Chapter 1: Mathematics of Merchandising

Profit = Sales – Costs

Gross Profit $= Sales -$ Cost of Goods Sold $= Sales - COGS$

Net Profit $= Sales -$ Total Costs $= GP -$ Operating Expenses

Percent Markup $= \dfrac{Profit}{Cost}$

Percent Margin $= \dfrac{Profit}{Sales}$

$Sales = Cost \times (1+$

$Net = List(1 - d_1)(1 - d_2)(1 - d_3)$

Chapter 2: Functions

$$Slope = \frac{Rise}{Run} = \frac{y_2 - y_1}{x_2 - x_1}$$

Equation of a line: $Y = Intercept + slope \times X$

$Revenue = price \times Quantity = price \times X$

$Costs =$ Fixed Costs $-$ Variable Costs $\times\ Quantity = FC + VCX$

$Profit = Revenue - Costs = price \times X - FC - VCX = (price - VC)X - FC$

Contribution Margin $= price - VC =$ markup in dollars for 1 item

Chapter 3: Simple Interest

$$I = Prt$$

$$FV = P + I = P(1 + rt)$$

$$P = \frac{FV}{1 + rt}$$

Chapter 4: Compound Interest

$$FV = PV(1 + i)^n$$

$$i = \frac{j_m}{m}$$

$$PV = \frac{FV}{(1 + i)^n} = FV(1 + i)^(- n)$$

Chapter 5: Annuities

Ordinary Perpetuities:

$$PMT = PV \times i$$

$$PV = \frac{PMT}{i}$$

Perpetuities Due

$$PMT_{Due} = \frac{PV \times i}{1 + i}$$

$$PV_{Due} = \frac{PMT}{i} + PMT$$

To switch to payments at the beginning of the interval:

- Press **2ND BGN** (above the PMT key). The display should show END.
- Press **2ND SET** (above the ENTER key). The display should show BGN.

- Press **CE/C** (bottom left corner) or **2ND QUIT** (top left corner).

To go back to payments at the END of the interval:

- Press **2ND BGN** (above the PMT key) The display should show BGN.
- Press **2ND SET** (above the ENTER key). The display should show END.
- Press **CE/C** (bottom left corner) or **2ND QUIT** (top left corner)

Chapter 6: Investment Decisions